

QUIVERS WITH RELATIONS FOR SYMMETRIZABLE CARTAN MATRICES III: CONVOLUTION ALGEBRAS

CHRISTOF GEISS, BERNARD LECLERC, AND JAN SCHRÖER

ABSTRACT. We realize the enveloping algebra of the positive part of a semisimple complex Lie algebra as a convolution algebra of constructible functions on module varieties of some Iwanaga-Gorenstein algebras of dimension 1.

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1. INTRODUCTION

1.1. Let Q be a finite quiver without oriented cycles. Let C be the symmetric generalized Cartan matrix corresponding to the undirected graph underlying Q , and let $\mathfrak{g} = \mathfrak{g}(C)$ be the Kac-Moody Lie algebra attached to C . Kac [K1] has shown that the dimension vectors of the indecomposable representations of Q form the roots of the positive part $\mathfrak{n} = \mathfrak{n}(C)$ of \mathfrak{g} . For quivers Q of finite type, Ringel [Rin2, Rin3] found a direct construction of the Lie algebra \mathfrak{n} itself, and of its enveloping algebra $U(\mathfrak{n})$, in terms of the representation theory of Q . He used Hall polynomials counting extensions of representations over \mathbb{F}_q , and recovered $U(\mathfrak{n})$ as the Hall algebra of the path algebra $\mathbb{F}_q Q$ specialized at $q = 1$. Later, Schofield [S] replaced counting points of varieties over \mathbb{F}_q by taking the Euler characteristic of complex varieties, and extended Ringel's result to an arbitrary quiver (see also [Rie] in the finite type case). Finally, Lusztig [L1] reformulated Schofield's construction and obtained $U(\mathfrak{n})$ as a convolution algebra of constructible functions over the affine spaces $\text{rep}(Q, \mathbf{d})$ of representations of $\mathbb{C}Q$ with dimension vector \mathbf{d} .

This paper is a first step towards a broad generalization of Schofield's theorem. We take for C an arbitrary symmetrizable generalized Cartan matrix [K2, Sections 1.1 and 2.1]. This means that there exists a diagonal matrix D with positive integer diagonal entries such that DC is symmetric. The corresponding Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(C)$ is called symmetrizable. With this datum together with an orientation Ω of the graph naturally attached to C , we have associated in [GLS1] a finite-dimensional algebra $H = H(C, D, \Omega)$ defined by a quiver with relations. This algebra makes sense over an arbitrary field K , but here we fix $K = \mathbb{C}$ so that varieties of H -modules are complex varieties. When

C is symmetric and D is the identity matrix, H is just the path algebra of the quiver Q corresponding to C and Ω . In general, it is shown in [GLS1] that H is Iwanaga-Gorenstein of dimension 1, and that its category of locally free modules (*i.e.* modules of homological dimension ≤ 1) carries an Euler form whose symmetrization is given by DC . By [GLS2, Proposition 3.1] the affine varieties $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ of locally free H -modules with rank vector \mathbf{r} are smooth and irreducible. By analogy with [S, L1], we then introduce a convolution bialgebra $\mathcal{M} = \mathcal{M}(H)$ of constructible functions on the varieties $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$. We show that \mathcal{M} is a Hopf algebra isomorphic to the enveloping algebra of the Lie algebra of its primitive elements (Proposition 4.7). Let again \mathfrak{n} denote the positive part of the symmetrizable Kac-Moody Lie algebra \mathfrak{g} . Our main result is then:

Theorem 1.1. *For $H = H(C, D, \Omega)$, $\mathcal{M} = \mathcal{M}(H)$ and $\mathfrak{n} = \mathfrak{n}(C)$ the following hold:*

(i) *There is a surjective Hopf algebra homomorphism*

$$\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}.$$

(ii) *Assume that C is of Dynkin type. Then η_H is a Hopf algebra isomorphism.*

We conjecture that η_H is an isomorphism for all symmetrizable generalized Cartan matrices C and all symmetrizers D .

This generalizes Schofield's theorem in two directions. Firstly, Theorem 1.1 gives a new complex geometric construction of $U(\mathfrak{n})$ for non-symmetric Cartan matrices of Dynkin type. Secondly, note that if D is a symmetrizer for C , then kD is also a symmetrizer for every $k \geq 1$. As k increases, the categories of locally free modules over $H(C, kD, \Omega)$ become more and more rich and complicated, the dimension of the varieties $\text{rep}_{\text{l.f.}}(H(C, kD, \Omega), \mathbf{r})$ increase and their orbit structure gets finer, but at least in the Dynkin case, and conjecturally in all cases, the convolution algebras $\mathcal{M}(H(C, kD, \Omega))$ remain the same. Thus for every semisimple complex Lie algebra \mathfrak{g} we get an infinite series of exact categories $\text{rep}_{\text{l.f.}}(H(C, kD, \Omega))$, ($k \geq 1$) whose convolution algebras $\mathcal{M}(H(C, kD, \Omega))$ are isomorphic to $U(\mathfrak{n})$. This appears to be new, even when C is symmetric.

Theorem 1.1 is inspired from [S]. In fact when C is symmetric and D is the identity matrix the algebra \mathcal{M} coincides with the algebra denoted by $R^+(Q)$ in [S]. However in all other cases, \mathcal{M} is defined using filtrations which are not composition series. We also note that, following [L1] and in contrast to [S], we systematically use the language of constructible functions and convolution products.

1.2. Let us outline the structure of the paper. In Section 2 we briefly recall the definition of a representation of a modulated graph in the sense of Dlab and Ringel. In Section 3 we review the definition of $H(C, D, \Omega)$ and the results of [GLS1] which will be needed in the sequel. In Section 4 we introduce the algebra of constructible functions $\mathcal{M}(H)$ and show that it is a homomorphic image of $U(\mathfrak{n})$ (Corollary 4.10). In Section 5 we construct for the preprojective modules over the algebras $H = H(C, D, \Omega)$ analogues of the Auslander-Reiten sequences of the preprojective representations of a modulated graph associated with the datum (C, D, Ω) . This will be used in the proof of our main result, but it should also be of independent interest. Section 6 is devoted to the explicit construction of certain primitive elements in the Hopf algebra $\mathcal{M}(H)$ in the case where C is of type B_n , C_n , F_4 or G_2 , and D is the minimal symmetrizer. In Section 7 we prove that the homomorphism $U(\mathfrak{n}) \rightarrow \mathcal{M}(H)$ is an isomorphism provided C is of Dynkin type. For a minimal symmetrizer D , this follows from Section 6. For the case of an arbitrary symmetrizer, we use a geometric result from [GLS2]. Finally, we discuss some examples in Section 8.

1.3. Notation. By \mathbb{N} we denote the natural numbers including 0. If not mentioned otherwise, by a *module* we mean a finite-dimensional left module. For a module M and a positive integer m , let M^m be the direct sum of m copies of M .

2. REPRESENTATIONS OF MODULATED GRAPHS

Following Dlab and Ringel [DR] we quickly review the definition of a representation of a modulated graph. Note however that [DR] uses somewhat dual conventions than those presented here.

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a *symmetrizable generalized Cartan matrix*, and let $D = \text{diag}(c_1, \dots, c_n)$ be a *symmetrizer* of C . This means that $c_i \in \mathbb{Z}_{>0}$, and

$$c_{ii} = 2, \quad c_{ij} \leq 0 \quad \text{for} \quad i \neq j, \quad c_i c_{ij} = c_j c_{ji}.$$

An *orientation* of C is a subset $\Omega \subset \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ such that the following hold:

- (i) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
- (ii) For each sequence $((i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we have $i_1 \neq i_{t+1}$.

Let F be a field, and let $(F_i, {}_i F_j)$ with $1 \leq i \leq n$ and $(i, j) \in \Omega$ be a *modulation* associated with (C, D, Ω) . Thus F_i is an F -skew-field with $\dim_F(F_i) = c_i$, and ${}_i F_j$ is an F_i - F_j -bimodule such that F acts centrally on ${}_i F_j$, and we have ${}_i F_j \cong F_i^{|c_{ij}|}$ as left F_i -modules and ${}_i F_j \cong F_j^{|c_{ji}|}$ as right F_j -modules. (Such a modulation exists, provided we work over a suitable ground field F .) A *representation* (X_i, X_{ij}) of this modulation consists of a finite-dimensional F_i -module X_i for each $1 \leq i \leq n$, and of an F_i -linear map

$$X_{ij}: {}_i F_j \otimes_{F_j} X_j \rightarrow X_i$$

for each $(i, j) \in \Omega$. Let

$$S := \prod_{i=1}^n F_i \quad \text{and} \quad B := \bigoplus_{(i,j) \in \Omega} {}_i F_j.$$

Thus B is an S - S -bimodule. Let $T = T(C, D, \Omega) = T_S(B)$ be the corresponding tensor algebra. The algebra T is a finite-dimensional hereditary F -algebra, and the abelian category of representations of the modulation $(F_i, {}_i F_j)$ is equivalent to the category $\text{mod}(T)$ of finite-dimensional T -modules. We refer to [DR] for further details on the representation theory of modulated graphs.

3. THE ALGEBRAS $H(C, D, \Omega)$

3.1. Definition of $H(C, D, \Omega)$. We use the same notation as in [GLS1]. Let (C, D, Ω) be as in Section 2, i.e. C is a symmetrizable generalized Cartan matrix, $D = \text{diag}(c_1, \dots, c_n)$ is a symmetrizer and Ω an orientation of C . When $c_{ij} < 0$ define

$$g_{ij} := |\gcd(c_{ij}, c_{ji})|, \quad f_{ij} := |c_{ij}|/g_{ij}.$$

Let $Q := Q(C, \Omega) := (Q_0, Q_1)$ be the quiver with vertex set $Q_0 := \{1, \dots, n\}$ and with arrow set

$$Q_1 := \{\alpha_{ij}^{(g)}: j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i: i \rightarrow i \mid 1 \leq i \leq n\}.$$

Let

$$H := H(C, D, \Omega) := \mathbb{C}Q/I$$

where $\mathbb{C}Q$ is the path algebra of Q , and I is the ideal of $\mathbb{C}Q$ defined by the following relations:

(H1) for each i , we have

$$\varepsilon_i^{c_i} = 0;$$

(H2) for each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$, we have

$$\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.$$

This definition is illustrated by many examples in [GLS1, Section 13].

Clearly, H is a finite-dimensional \mathbb{C} -algebra. It is known [GLS1, Theorem 1.1] that H is Iwanaga-Gorenstein of dimension 1. This means that for an H -module M the following are equivalent:

- $\text{proj. dim}(M) < \infty$,
- $\text{inj. dim}(M) < \infty$,
- $\text{proj. dim}(M) \leq 1$,
- $\text{inj. dim}(M) \leq 1$.

Moreover, if M is a submodule of a projective H -module and if $\text{proj. dim}(M) \leq 1$, then M is projective. Dually, if M is a quotient module of an injective H -module and $\text{inj. dim}(M) \leq 1$ then M is injective.

Note that if C is symmetric and if D is the identity matrix, then H is isomorphic to the path algebra $\mathbb{C}Q^\circ$, where Q° is the acyclic quiver obtained from Q by deleting all loops ε_i . More generally, it is easy to see that if C is symmetric and $D = \text{diag}(k, \dots, k)$ for some $k > 0$, then H is isomorphic to $RQ^\circ := R \otimes_{\mathbb{C}} \mathbb{C}Q^\circ$, where R is the truncated polynomial ring $\mathbb{C}[x]/(x^k)$. In that case, H -modules are nothing else than representations of Q° over the ring R . When C is only symmetrizable, one has a similar picture by replacing the path algebra RQ° by a generalized modulated graph over a family of truncated polynomial rings, as we shall now explain.

3.2. Generalized modulated graphs. Let $H = H(C, D, \Omega)$. It was shown in [GLS1, Section 5] that H gives rise to a generalized modulated graph, and that the category of H -modules is isomorphic to the category of representations of this generalized modulated graph. This viewpoint, which is very close to Dlab and Ringel's [DR] representation theory of modulated graphs outlined in Section 2, will be useful in several places below.

For $i = 1, \dots, n$, let H_i be the subalgebra of H generated by ε_i . Thus H_i is isomorphic to $\mathbb{C}[x]/(x^{c_i})$. For $(i, j) \in \Omega$, we define

$${}_iH_j := H_i \text{Span}_{\mathbb{C}}(\alpha_{ij}^{(g)} \mid 1 \leq g \leq g_{ij}) H_j.$$

It is shown in [GLS1] that ${}_iH_j$ is an H_i - H_j -bimodule, which is free as a left H_i -module and free as a right H_j -module. An H_i -basis of ${}_iH_j$ is given by

$$\{\alpha_{ij}^{(g)}, \alpha_{ij}^{(g)} \varepsilon_j, \dots, \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}-1} \mid 1 \leq g \leq g_{ij}\}.$$

In particular, we have an isomorphism ${}_iH_j \cong H_i^{|c_{ij}|}$ of left H_i -modules, and we have an isomorphism ${}_iH_j \cong H_j^{|c_{ji}|}$ of right H_j -modules.

The tuple $(H_i, {}_iH_j)$ with $1 \leq i \leq n$ and $(i, j) \in \Omega$ is called a *generalized modulation* associated with the datum (C, D, Ω) . A *representation* (M_i, M_{ij}) of this generalized modulation consists of a finite-dimensional H_i -module M_i for each $1 \leq i \leq n$, and of an H_i -linear map

$$M_{ij}: {}_iH_j \otimes_{H_j} M_j \rightarrow M_i$$

for each $(i, j) \in \Omega$. The representations of this generalized modulation form an abelian category $\text{rep}(C, D, \Omega)$ isomorphic to the category of H -modules [GLS1, Proposition 5.1]. (Here we identify the category of H -modules with the category of representations of the quiver $Q(C, \Omega)$ satisfying the relations (H1) and (H2).) Given a representation (M_i, M_{ij}) in $\text{rep}(C, D, \Omega)$ the corresponding H -module $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is obtained as follows: the \mathbb{C} -linear map $M(\varepsilon_i): M_i \rightarrow M_i$ is given by

$$M(\varepsilon_i)(m) := \varepsilon_i m.$$

(Here we use that M_i is an H_i -module). For $(i, j) \in \Omega$, the \mathbb{C} -linear map $M(\alpha_{ij}^{(g)}): M_j \rightarrow M_i$ is defined by

$$M(\alpha_{ij}^{(g)})(m) := M_{ij}(\alpha_{ij}^{(g)} \otimes m).$$

The maps $M(\alpha_{ij}^{(g)})$ and $M(\varepsilon_i)$ satisfy the defining relations (H1) and (H2) of H because the maps M_{ij} are H_i -linear.

3.3. Locally free H -modules. We say that an H -module $M = (M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is *locally free* if for every i the H_i -module M_i is free. By [GLS1, Theorem 1.1], M is locally free if and only if $\text{proj. dim}(M) \leq 1$. The full subcategory $\text{rep}_{\text{l.f.}}(H)$ whose objects are the finite-dimensional locally free modules is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms, and it has Auslander-Reiten sequences [GLS1, Lemma 3.8, Theorem 3.9].

The *rank vector* of $M \in \text{rep}_{\text{l.f.}}(H)$ is defined as $\underline{\text{rank}}(M) = (\text{rk}(M_1), \dots, \text{rk}(M_n))$. Let $\alpha_1, \dots, \alpha_n$ be the standard basis of \mathbb{Z}^n . For $1 \leq i \leq n$ we denote by E_i the unique locally free H -module with rank vector α_i . In other words, E_i is nothing else than H_i regarded as an H -module in the obvious way.

For $M, N \in \text{rep}_{\text{l.f.}}(H)$, the integer

$$\langle M, N \rangle_H := \dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N)$$

depends only on the rank vectors $\underline{\text{rank}}(M)$ and $\underline{\text{rank}}(N)$, see [GLS1, Proposition 4.1]. The map $(M, N) \mapsto \langle M, N \rangle_H$ thus descends to a bilinear form on the Grothendieck group \mathbb{Z}^n of $\text{rep}_{\text{l.f.}}(H)$, given on the basis $\alpha_i = \underline{\text{rank}}(E_i)$ by

$$\langle \alpha_i, \alpha_j \rangle_H = \begin{cases} c_i c_{ij} & \text{if } (j, i) \in \Omega, \\ c_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(-, -)_H$ be the symmetrization of $\langle -, - \rangle_H$ defined by $(a, b)_H := \langle a, b \rangle_H + \langle b, a \rangle_H$, and let q_H be the quadratic form defined by $q_H(a) := \langle a, a \rangle_H$. Note that $(-, -)_H$ is nothing else than the symmetric bilinear form

$$(\alpha_i, \alpha_j) = c_i c_{ij}, \quad (1 \leq i, j \leq n)$$

associated with the symmetric matrix DC .

4. THE CONVOLUTION ALGEBRA \mathcal{M}

4.1. Definition of the algebra \mathcal{M} . As before, let $H = H(C, D, \Omega)$ and $Q = Q(C, \Omega)$. For an arrow $a: i \rightarrow j$ in Q set $s(a) = i$ and $t(a) = j$. Let $\text{rep}(H, \mathbf{d})$ be the affine complex variety of H -modules with dimension vector $\mathbf{d} = (d_1, \dots, d_n)$. By definition the closed points in $\text{rep}(H, \mathbf{d})$ are tuples

$$M = (M(a))_{a \in Q_1} \in \prod_{a \in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$$

of \mathbb{C} -linear maps such that

$$M(\varepsilon_i)^{c_i} = 0$$

and for each $(i, j) \in \Omega$ and $1 \leq g \leq g_{ij}$ we have

$$M(\varepsilon_i)^{f_{ji}} M(\alpha_{ij}^{(g)}) = M(\alpha_{ij}^{(g)}) M(\varepsilon_j)^{f_{ij}}.$$

The group $G_{\mathbf{d}} := \text{GL}_{d_1} \times \dots \times \text{GL}_{d_n}$ acts on $\text{rep}(H, \mathbf{d})$ by conjugation. The $G_{\mathbf{d}}$ -orbit of $M \in \text{rep}(H, \mathbf{d})$ is denoted by \mathcal{O}_M . The $G_{\mathbf{d}}$ -orbits of $\text{rep}(H, \mathbf{d})$ are in one-to-one correspondence with isomorphism classes of H -modules with dimension vector \mathbf{d} .

Recall that a *constructible function* on a complex algebraic variety V is a map $\varphi: V \rightarrow \mathbb{C}$ such that the image of φ is finite, and for each $a \in \mathbb{C}$ the preimage $\varphi^{-1}(a)$ is a constructible subset of V . Let $\mathcal{F}_{\mathbf{d}}$ be the complex vector space of constructible functions $f: \text{rep}(H, \mathbf{d}) \rightarrow \mathbb{C}$ which are constant on $G_{\mathbf{d}}$ -orbits, and let

$$\mathcal{F} = \mathcal{F}(H) = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{F}_{\mathbf{d}}.$$

We endow \mathcal{F} with a convolution product $*$ defined as in [L1, Section 10.12] or [L2, Section 2.1], using Euler characteristics of constructible subsets. Namely, we put

$$(f * g)(X) = \int_{Y \subseteq X} f(Y) g(X/Y) d\chi, \quad (f, g \in \mathcal{F}, X \in \text{rep}(H)).$$

Here, the integral is taken on the projective variety of all H -submodules Y of X , and for a constructible function φ on a variety V , we set

$$\int_{Y \in V} \varphi(Y) d\chi = \sum_{a \in \mathbb{C}} a \cdot \chi(\varphi^{-1}(a)).$$

It is well known that $(\mathcal{F}, *)$ has the structure of an \mathbb{N}^n -graded associative \mathbb{C} -algebra, see e.g. [BT, Section 4.2].

Let $\mathbf{e}_i = (0, \dots, c_i, \dots, 0) \in \mathbb{N}^n$ be the dimension vector of E_i . Let $\theta_i \in \mathcal{F}$ denote the characteristic function of the $G_{\mathbf{e}_i}$ -orbit of $\text{rep}(H, \mathbf{e}_i)$ corresponding to E_i . In other words, we have

$$\theta_i(M) = \begin{cases} 1 & \text{if } M \cong E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.1. We denote by $\mathcal{M} = \mathcal{M}(H)$ the subalgebra of $(\mathcal{F}, *)$ generated by $\theta_1, \dots, \theta_n$, and for $\mathbf{d} \in \mathbb{N}^n$ we set $\mathcal{M}_{\mathbf{d}} = \mathcal{M} \cap \mathcal{F}_{\mathbf{d}}$.

Note that $\mathcal{M}_{\mathbf{d}}$ is a finite-dimensional vector space. The identity element $\mathbf{1}_{\mathcal{M}}$ of \mathcal{M} is the characteristic function of the zero H -module.

Lemma 4.2. *Let $f \in \mathcal{M}_{\mathbf{d}}$ and $X \in \text{rep}(H, \mathbf{d})$. If X is not locally free, then $f(X) = 0$.*

Proof. For an H -module X and a sequence $\mathbf{i} = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$, we have, by definition of the convolution product $*$,

$$(\theta_{i_1} * \dots * \theta_{i_k})(X) = \chi(\mathcal{FL}_{X, \mathbf{i}}), \quad (4.1)$$

where $\mathcal{FL}_{X, \mathbf{i}}$ is the constructible set of flags

$$(0 = X_0 \subset X_1 \subset \dots \subset X_k = X)$$

of H -modules such that $X_j/X_{j-1} \cong E_{i_j}$ for all $1 \leq j \leq k$. By [GLS1, Lemma 3.8] the category of locally free H -modules is closed under extensions, hence if X is not locally free we have $\mathcal{FL}_{X, \mathbf{i}} = \emptyset$ for every sequence \mathbf{i} . This shows that $(\theta_{i_1} * \dots * \theta_{i_k})(X) = 0$ for every sequence \mathbf{i} , and thus, by definition of \mathcal{M} , that $f(X) = 0$ for every $f \in \mathcal{M}$. \square

Remark 4.3. When the Cartan matrix C is symmetric and D is the identity matrix, the algebra H is the path algebra $\mathbb{C}Q^\circ$ (see Section 3.1). In that case $\mathcal{M}(H)$ coincides with the algebra $R^+(\mathbb{C}Q^\circ)$ of [S, Section 2.3], and with the algebra $\mathcal{M}_0(\Omega)$ of [L1, Section 10.19].

4.2. Varieties of locally free H -modules. Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ be a dimension vector. If $M \in \text{rep}(H, \mathbf{d})$ is locally free, its rank vector is $\mathbf{r} = (r_1, \dots, r_n)$ where $r_i = d_i/c_i$. Hence locally free modules can only exist if d_i is divisible by c_i for every i . In this case we say that \mathbf{d} is \mathbf{c} -divisible. Let $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ be the union of all $G_{\mathbf{d}}$ -orbits \mathcal{O}_M of locally free modules M with rank vector \mathbf{r} . By Lemma 4.2, the support of every constructible function $f \in \mathcal{M}_{\mathbf{d}}$ is contained in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$.

Proposition 4.4 ([GLS2, Proposition 3.1]). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be \mathbf{c} -divisible as above. Set $r_i := d_i/c_i$ and $\mathbf{r} = (r_1, \dots, r_n)$. Then $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ is a non-empty open subset of $\text{rep}(H, \mathbf{d})$. Furthermore, $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ is smooth and irreducible of dimension $\dim(G_{\mathbf{d}}) - q_H(\mathbf{r})$.*

4.3. Bialgebra structure of \mathcal{M} . Consider the direct product of algebras $H \times H$. Modules for $H \times H$ are pairs (X_1, X_2) of modules for H . An $H \times H$ -submodule of (X_1, X_2) is a pair (Y_1, Y_2) where Y_1 is an H -submodule of X_1 and Y_2 is an H -submodule of X_2 . Note that we can regard $H \times H$ as the algebra $H(C \oplus C, D \oplus D, \Omega \oplus \Omega)$, where $C \oplus C$ (resp. $D \oplus D$) means the block diagonal matrix with two diagonal blocks equal to C (resp. D), and $\Omega \oplus \Omega$ is the obvious orientation of $C \oplus C$ induced by the orientation Ω of C . Therefore we can define as above a convolution algebra $\mathcal{F}(H \times H)$.

We have an algebra embedding of $\mathcal{F}(H) \otimes \mathcal{F}(H)$ into $\mathcal{F}(H \times H)$ by setting

$$(f \otimes g)(X, Y) = f(X)g(Y).$$

Following Ringel [Rin3] (see also [BT, Section 4.3]), one introduces a map

$$c: \mathcal{F}(H) \rightarrow \mathcal{F}(H \times H)$$

by $c(f)(X, Y) = f(X \oplus Y)$.

Proposition 4.5. *The map c restricts to an algebra homomorphism*

$$c: \mathcal{M}(H) \rightarrow \mathcal{M}(H) \otimes \mathcal{M}(H)$$

for the convolution product such that

$$c(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

This makes $\mathcal{M}(H)$ into a cocommutative bialgebra.

Proof. We first show that c is a homomorphism from $\mathcal{F}(H)$ to $\mathcal{F}(H \times H)$ for the convolution product. Indeed, on the one hand we have for $f, g \in \mathcal{F}(H)$

$$(c(f * g))(X, Y) = \int_{Z \subseteq X \oplus Y} f(Z) g((X \oplus Y)/Z) d\chi, \quad (4.2)$$

and on the other hand

$$(c(f) * c(g))(X, Y) = \int_{Z_1 \subseteq X, Z_2 \subseteq Y} f(Z_1 \oplus Z_2) g((X \oplus Y)/(Z_1 \oplus Z_2)) d\chi. \quad (4.3)$$

To show that the two integrals are the same, we consider the \mathbb{C}^* -action on $X \oplus Y$ given by

$$\lambda \cdot (x, y) = (\lambda x, y), \quad (x \in X, y \in Y, \lambda \in \mathbb{C}^*).$$

This induces a \mathbb{C}^* -action on the variety of submodules Z of $X \oplus Y$, whose fixed points are exactly the submodules of the form $Z = Z_1 \oplus Z_2$ with $Z_1 \subseteq X$ and $Z_2 \subseteq Y$. Moreover, for a submodule Z of $X \oplus Y$ and $\lambda \in \mathbb{C}^*$, the H -module $\lambda \cdot Z$ is isomorphic to Z , so for every $f \in \mathcal{F}(H)$ we have $f(\lambda \cdot Z) = f(Z)$. Thus, using [BB, Corollary 2], we get that (4.2) and (4.3) are equal. It follows that c restricts to an algebra homomorphism $\mathcal{M}(H) \rightarrow \mathcal{F}(H \times H)$. Since E_i is indecomposable, we have

$$c(\theta_i)(X, Y) = \theta_i(X \oplus Y) = \begin{cases} 1 & \text{if } X \cong E_i \text{ and } Y = 0, \text{ or } X = 0 \text{ and } Y \cong E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $c(\theta_i)$ can be identified with $\theta_i \otimes 1 + 1 \otimes \theta_i \in \mathcal{M}(H) \otimes \mathcal{M}(H) \subset \mathcal{F}(H) \otimes \mathcal{F}(H) \subset \mathcal{F}(H \times H)$. Finally, since $\mathcal{M}(H)$ is generated by the θ_i 's and c is multiplicative, this implies that the image $c(\mathcal{M}(H))$ is indeed contained in $\mathcal{M}(H) \otimes \mathcal{M}(H)$. \square

An element f of \mathcal{M} is called *primitive* if $c(f) = f \otimes 1 + 1 \otimes f$.

Lemma 4.6. *An element f of \mathcal{M} is primitive if and only if f is supported only on indecomposable modules.*

Proof. This follows immediately from the equality $f(X \oplus Y) = c(f)(X, Y)$. \square

It is easy to see that if f and g are primitive, then the Lie bracket

$$[f, g] := f * g - g * f$$

is also primitive. Hence the subspace $\mathcal{P}(\mathcal{M}) \subset \mathcal{M}$ of primitive elements has a natural Lie algebra structure.

Proposition 4.7. *$(\mathcal{M}, *, c)$ is a Hopf algebra isomorphic to the universal enveloping algebra $U(\mathcal{P}(\mathcal{M}))$.*

Proof. A nonzero element f of \mathcal{M} is *group-like* if $c(f) = f \otimes f$. Arguing as in [BT, Section 4.5], we see that the only group-like element is the identity element $1_{\mathcal{M}}$. Indeed, if f is group-like for any H -module X and $k \in \mathbb{N}$ we have $f(X^k) = f(X)^k$. If $f(X) \neq 0$ for a module $X \neq 0$, then the decomposition of f with respect to the direct sum $\bigoplus_{\mathbf{d}} \mathcal{M}_{\mathbf{d}}$ has infinitely many nonzero components, a contradiction. Hence we have $f = \lambda 1_{\mathcal{M}}$ for some scalar $\lambda \neq 0$. Since $f(0 \oplus 0) = f(0)f(0)$, we get that $\lambda = \lambda^2$. This implies $f = 1_{\mathcal{M}}$.

Therefore, we can repeat the last part of the proof of [Rin3, Theorem]: Since \mathcal{M} is a cocommutative coalgebra over \mathbb{C} , [Sw, Lemma 8.0.1] together with the previous paragraph implies that \mathcal{M} is irreducible, hence a Hopf algebra [Sw, Theorem 9.2.2]. It then follows from [Sw, Theorem 13.0.1] that \mathcal{M} is isomorphic as a Hopf algebra to the universal enveloping algebra $U(\mathcal{P}(\mathcal{M}))$ of the Lie algebra $\mathcal{P}(\mathcal{M})$. \square

Remark 4.8. When the Cartan matrix C is symmetric and D is the identity matrix, the Lie algebra $\mathcal{P}(\mathcal{M})$ coincides with the Lie algebra $L^+(\mathbb{C}Q^\circ) = L^+(Q^\circ)$ of [S, Section 2.6].

4.4. Relations in \mathcal{M} . For $f \in \mathcal{M}$ we denote by $\text{ad } f$ the endomorphism of \mathcal{M} defined by

$$\text{ad } f(g) := [f, g], \quad (g \in \mathcal{M}).$$

Proposition 4.9. *The generators $\theta_1, \dots, \theta_n$ of \mathcal{M} satisfy the relations*

$$(\text{ad } \theta_i)^{1-c_{ij}}(\theta_j) = 0$$

for all $i \neq j$.

Proof. Since $\theta_i \in \mathcal{P}(\mathcal{M})$, we have

$$\Theta_{ij} := (\text{ad } \theta_i)^{1-c_{ij}}(\theta_j) \in \mathcal{P}(\mathcal{M})$$

for all $i \neq j$. By Lemma 4.2 and Lemma 4.6, to check that $\Theta_{ij} = 0$ it is therefore sufficient to check that there is no indecomposable locally free H -module with rank vector $(1 - c_{ij})\alpha_i + \alpha_j$. Let M be a locally free module M with this rank vector.

Let us assume first that $(i, j) \in \Omega$. Then, by Section 3.2, M is given by an H_i -linear map

$$M_{ij}: {}_iH_j \otimes_{H_j} M_j \rightarrow M_i,$$

where $M_j = H_j$ and $M_i = H_i^{1-c_{ij}}$. Now, ${}_iH_j \otimes_{H_j} M_j \cong {}_iH_j$ is a free H_i -module of rank $-c_{ij}$, so M_i contains a direct summand N_i isomorphic to H_i such that $N_i \cap \text{Im}(M_{ij}) = 0$. It follows that M has a direct summand isomorphic to E_i , and therefore M is not indecomposable.

For $(j, i) \in \Omega$, M is given by an H_j -linear map

$$M_{ji}: {}_jH_i \otimes_{H_i} M_i \rightarrow M_j.$$

Let

$$M_{ji}^\vee: M_i \rightarrow {}_iH_j \otimes_{H_j} M_j$$

be the associated adjoint map as defined in [GLS1, Section 5]. The map M_{ji}^\vee is H_i -linear, and as H_i -modules we have $M_i = H_i^{1-c_{ij}}$ and ${}_iH_j \otimes_{H_j} M_j \cong H_i^{-c_{ij}}$. Similarly as before, M_i contains a direct summand N_i isomorphic to H_i such that $N_i \cap \text{Ker}(M_{ji}^\vee) = 0$. It follows that M has a direct summand isomorphic to E_i . So again, M is not indecomposable. \square

Let $\mathfrak{g} = \mathfrak{g}(C)$ be the symmetrizable Kac-Moody Lie algebra over \mathbb{C} with Cartan matrix C . It is defined by the following presentation. There are $3n$ generators e_i, f_i, h_i ($1 \leq i \leq n$) subject to the relations:

- (i) $[e_i, f_j] = \delta_{ij} h_i$;
- (ii) $[h_i, h_j] = 0$;
- (iii) $[h_i, e_j] = c_{ij} e_j, [h_i, f_j] = -c_{ij} f_j$;
- (iv) $(\text{ad } e_i)^{1-c_{ij}}(e_j) = 0, (\text{ad } f_i)^{1-c_{ij}}(f_j) = 0 \quad (i \neq j)$.

Let $\mathfrak{n} = \mathfrak{n}(C)$ be the Lie subalgebra generated by e_i ($1 \leq i \leq n$). Then $U(\mathfrak{n})$ is the associative \mathbb{C} -algebra with generators e_i ($1 \leq i \leq n$) subject to the relations

$$(\text{ad } e_i)^{1-c_{ij}}(e_j) = 0, \quad (1 \leq i \neq j \leq n).$$

Corollary 4.10. *The assignment $e_i \mapsto \theta_i$ extends to a surjective Hopf algebra homomorphism*

$$\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}.$$

Proof. The algebras $U(\mathfrak{n})$ and \mathcal{M} are generated as algebras by their subsets \mathfrak{n} and $\mathcal{P}(\mathcal{M})$ of primitive elements, respectively. It follows from Proposition 4.9 that $e_i \mapsto \theta_i$ extends to a surjective algebra homomorphism $\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}$. The comultiplication of $U(\mathfrak{n})$ and \mathcal{M} are given by $e_i \mapsto e_i \otimes 1 + 1 \otimes e_i$ and $\theta_i \mapsto \theta_i \otimes 1 + 1 \otimes \theta_i$, respectively. The antipodes of $U(\mathfrak{n})$ and \mathcal{M} are given by $x \mapsto -x$ and $m \mapsto -m$ for all $x \in \mathfrak{n}$ and all $m \in \mathcal{P}(\mathcal{M})$, respectively. It follows that η_H is a Hopf algebra homomorphism. \square

4.5. Weight spaces. As before, let $\alpha_1, \dots, \alpha_n$ be the standard basis of \mathbb{Z}^n . As before, let $H = H(C, D, \Omega)$ and $\mathfrak{n} = \mathfrak{n}(C)$. The algebras $U(\mathfrak{n})$ and $\mathcal{M} = \mathcal{M}(H)$ are both \mathbb{N}^n -graded via $\deg(e_i) := \alpha_i$ and $\deg(\theta_i) := \alpha_i$, respectively. For $\alpha \in \mathbb{N}^n$ let $U(\mathfrak{n})_\alpha$ and \mathcal{M}_α be the vector spaces of elements of degree α . Define

$$\mathfrak{n}_\alpha := \mathfrak{n} \cap U(\mathfrak{n})_\alpha \quad \text{and} \quad \mathcal{P}(\mathcal{M})_\alpha := \mathcal{P}(\mathcal{M}) \cap \mathcal{M}_\alpha.$$

It follows that

$$\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{N}^n} \mathfrak{n}_\alpha \quad \text{and} \quad \mathcal{P}(\mathcal{M}) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{P}(\mathcal{M})_\alpha.$$

Furthermore, the surjective Hopf algebra homomorphism

$$\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}$$

restricts to a surjective Lie algebra homomorphism $\mathfrak{n} \rightarrow \mathcal{P}(\mathcal{M})$ and to a surjective linear map $\eta_{H,\alpha}: \mathfrak{n}_\alpha \rightarrow \mathcal{P}(\mathcal{M})_\alpha$ for each $\alpha \in \mathbb{N}^n$. The *weight space* \mathfrak{n}_α is non-zero if and only if α is a positive root of the Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(C)$. In particular, if C is a Cartan matrix of Dynkin type, then we have

$$\dim(\mathfrak{n}_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Delta^+(C), \\ 0 & \text{otherwise.} \end{cases}$$

5. PSEUDO AUSLANDER-REITEN SEQUENCES FOR PREPROJECTIVE MODULES

5.1. Auslander-Reiten translates and Coxeter transformation. Let $C \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix, let $D = \text{diag}(c_1, \dots, c_n)$ be a symmetrizer of C , and let Ω be an orientation of C .

Let $T = T(C, D, \Omega)$ be the tensor algebra of some modulation associated with (C, D, Ω) , compare Section 2. Up to isomorphism there are n simple T -modules S_1^T, \dots, S_n^T with $\dim \text{End}_T(S_i^T) = c_i$. For $1 \leq i \leq n$, let P_i^T (resp. I_i^T) be the indecomposable projective (resp. injective) T -module with $\text{top}(P_i^T) \cong S_i^T$ (resp. $\text{soc}(I_i^T) \cong S_i^T$). Let $c_T: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ be the Coxeter transformation of T . The automorphism c_T can be defined by the rule

$$c_T(\underline{\dim}(P_i^T)) = -\underline{\dim}(I_i^T)$$

for all $1 \leq i \leq n$.

Let $H = H(C, D, \Omega)$. Up to isomorphism there are n simple H -modules S_1^H, \dots, S_n^H corresponding to the vertices of $Q(C, D)$. For $1 \leq i \leq n$, let P_i^H (resp. I_i^H) be the indecomposable projective (resp. injective) H -module with $\text{top}(P_i^H) \cong S_i^H$ (resp. $\text{soc}(I_i^H) \cong S_i^H$). Let $c_H: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ be the Coxeter transformation of H , as defined in [GLS1, Section 2.5]. The automorphism c_H can be described by the rule

$$c_H(\underline{\text{rank}}(P_i^H)) = -\underline{\text{rank}}(I_i^H)$$

for all $1 \leq i \leq n$, see [GLS1, Section 3.4].

For $M, N \in \text{rep}_{\text{l.f.}}(H)$ and $X, Y \in \text{mod}(T)$, let

$$\langle M, N \rangle_H := \dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N)$$

and

$$\langle X, Y \rangle_T := \dim \operatorname{Hom}_T(X, Y) - \dim \operatorname{Ext}_T^1(X, Y).$$

Then [GLS1, Lemmas 3.2 and 3.3 and Section 4] imply the following crucial result.

Proposition 5.1. *We have*

- (i) $\underline{\dim}(P_i^T) = \underline{\operatorname{rank}}(P_i^H);$
- (ii) $\underline{\dim}(I_i^T) = \underline{\operatorname{rank}}(I_i^H);$
- (iii) $c_T = c_H;$
- (iv) *For $X, Y \in \operatorname{mod}(T)$ and $M, N \in \operatorname{rep}_{\text{l.f.}}(H)$ with $a = (a_1, \dots, a_n) = \underline{\dim}(X) = \underline{\operatorname{rank}}(M)$ and $b = (b_1, \dots, b_n) = \underline{\dim}(Y) = \underline{\operatorname{rank}}(N)$ we have*

$$\langle X, Y \rangle_T = \langle M, N \rangle_H = \sum_{i=1}^n c_i a_i b_i + \sum_{(j,i) \in \Omega} c_i c_{ij} a_i b_j.$$

By Proposition 5.1(iv) we can consider $\langle -, - \rangle_H$ and $\langle -, - \rangle_T$ as bilinear forms on $\mathbb{Z}^n \times \mathbb{Z}^n$.

Let τ_T (*resp.* τ_T^-) denote the Auslander-Reiten translation (*resp.* the inverse Auslander-Reiten translation) of the algebra T . By definition $\tau_T = \operatorname{DTr}$ is the dual of the transpose Tr . We refer to [ARS, Chapter 4] for further details. The next lemma follows from general Auslander-Reiten theory and from the fact that T is a hereditary algebra [Rin1, Section 4].

Lemma 5.2. *For non-projective indecomposable T -modules X and Y the following hold:*

- (i) $\operatorname{Hom}_T(X, Y) \cong \operatorname{Hom}_T(\tau_T(X), \tau_T(Y));$
- (ii) $\operatorname{Ext}_T^1(X, Y) \cong \operatorname{Ext}_T^1(\tau_T(X), \tau_T(Y));$
- (iii) $\underline{\dim}(\tau_T(X)) = c_T(\underline{\dim}(X)).$

There is an obvious dual of Lemma 5.2 for non-injective indecomposable T -modules X and Y .

Let τ_H (*resp.* τ_H^-) denote the Auslander-Reiten translation (*resp.* the inverse Auslander-Reiten translation) of the algebra H . Recall from [GLS1] that an indecomposable H -module M is called τ -locally free provided $\tau_H^m(M)$ is locally free for all $m \in \mathbb{Z}$.

Lemma 5.3. *For non-projective indecomposable τ -locally free H -modules M and N the following hold:*

- (i) $\operatorname{Hom}_H(M, N) \cong \operatorname{Hom}_H(\tau_H(M), \tau_H(N));$
- (ii) $\operatorname{Ext}_H^1(M, N) \cong \operatorname{Ext}_H^1(\tau_H(M), \tau_H(N));$
- (iii) $\underline{\operatorname{rank}}(\tau_H(M)) = c_H(\underline{\operatorname{rank}}(M)).$

Proof. Parts (i) and (iii) are a consequence of the results in [GLS1, Section 11.1]. Part (ii) follows then from [GLS1, Proposition 4.1]. \square

There is an obvious dual of Lemma 5.2 for non-injective indecomposable τ -locally free H -modules M and N .

5.2. Preprojective modules. As before, for $1 \leq i \leq n$ let P_i^T (*resp.* P_i^H) be the indecomposable projective T -module (*resp.* H -module) associated with i . Recall that an indecomposable T -module X (*resp.* an indecomposable H -module M) is *preprojective* if $X \cong \tau_T^{-k}(P_i^T)$ (*resp.* $M \cong \tau_H^{-k}(P_i^H)$) for some $1 \leq i \leq n$ and some $k \geq 0$. By [GLS1, Proposition 11.6], preprojective modules are locally free.

Proposition 5.4. *For all $k \geq 0$ we have*

$$\underline{\dim}(\tau_T^{-k}(P_i^T)) = \underline{\text{rank}}(\tau_H^{-k}(P_i^H)).$$

Proof. This follows immediately from Proposition 5.1 and Lemmas 5.2 and 5.3. \square

Proposition 5.5. *Let $M = \tau_H^{-k}(P_i^H)$ and $N = \tau_H^{-s}(P_j^H)$ be preprojective H -modules, and let $X = \tau_T^{-k}(P_i^T)$ and $Y = \tau_T^{-s}(P_j^T)$ be the corresponding preprojective T -modules. Then we have*

$$\begin{aligned} \dim \text{Hom}_H(M, N) &= \dim \text{Hom}_T(X, Y), \\ \dim \text{Ext}_H^1(M, N) &= \dim \text{Ext}_T^1(X, Y). \end{aligned}$$

Proof. By Proposition 5.1 we have $\langle \underline{\text{rank}}(M), \underline{\text{rank}}(N) \rangle_H = \langle \underline{\dim}(X), \underline{\dim}(Y) \rangle_T$. Since

$$\begin{aligned} \langle \underline{\text{rank}}(M), \underline{\text{rank}}(N) \rangle_H &= \dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N), \\ \langle \underline{\dim}(X), \underline{\dim}(Y) \rangle_T &= \dim \text{Hom}_T(X, Y) - \dim \text{Ext}_T^1(X, Y), \end{aligned}$$

it is enough to show that $\dim \text{Hom}_H(M, N) = \dim \text{Hom}_T(X, Y)$.

If $k \leq s$, then

$$\begin{aligned} \dim \text{Hom}_H(M, N) &= \dim \text{Hom}_H(P_i^H, \tau_H^{-(s-k)}(P_j^H)) \\ &= [\tau_H^{-(s-k)}(P_j^H) : S_i^H] \dim \text{End}_H(S_i^H) \\ &= [\tau_H^{-(s-k)}(P_j^H) : S_i^H] \\ &= \underline{\dim}(\tau_H^{-(s-k)}(P_j^H))_i \\ &= c_i(\underline{\text{rank}}(\tau_H^{-(s-k)}(P_j^H)))_i, \end{aligned}$$

and

$$\begin{aligned} \dim \text{Hom}_T(X, Y) &= \dim \text{Hom}_T(P_i^T, \tau_T^{-(s-k)}(P_j^T)) \\ &= [\tau_T^{-(s-k)}(P_j^T) : S_i^T] \dim \text{End}_T(S_i^T) \\ &= c_i(\underline{\dim}(\tau_T^{-(s-k)}(P_j^T)))_i. \end{aligned}$$

If $k > s$, then

$$\text{Hom}_H(M, N) \cong \text{Hom}_H(\tau_H^{-(k-s)}(P_i^H), P_j^H) = 0,$$

and

$$\text{Hom}_T(X, Y) \cong \text{Hom}_T(\tau_T^{-(k-s)}(P_i^T), P_j^T) = 0.$$

This finishes the proof. \square

Corollary 5.6. *Let $M = \tau_H^{-k}(P_i^H)$ and $N = \tau_H^{-s}(P_j^H)$ be preprojective H -modules. Then we have*

$$\begin{aligned} \dim \text{Hom}_H(M, N) &= c_i(\underline{\text{rank}}(\tau_H^{-(s-k)}(P_j^H)))_i, \\ \dim \text{Ext}_H^1(M, N) &= c_j(\underline{\text{rank}}(\tau_H^{-(k-s-1)}(P_i^H)))_j. \end{aligned}$$

Proof. The first equality follows from the proof of Proposition 5.5. The second equality follows from combining the first equality with the Auslander-Reiten formula

$$\dim \text{Ext}_H^1(M, N) = \dim \text{Hom}_H(\tau_H^{-1}(N), M).$$

(Since N is a module of injective dimension at most one, we don't need to consider stable homomorphism spaces.) \square

The dimension vectors (resp. rank vectors) of the preprojective T -modules (resp. H -modules) are positive roots of the Kac-Moody Lie algebra $\mathfrak{g}(C)$. (For H -modules, see [GLS1, Lemma 3.2 and Proposition 11.5].) A positive root $\alpha \in \Delta^+$ is *H-preprojective* if α is the dimension vector of a preprojective T -module (or equivalently the rank vector of a preprojective H -module). In this case let $X(\alpha)$ be the preprojective T -module with dimension vector α , and let $M(\alpha)$ be the preprojective H -module with rank vector α .

As a direct consequence of Proposition 5.5 we get the following result.

Corollary 5.7. *Let $\gamma_1, \dots, \gamma_t$ be H -preprojective positive roots. Then $X(\gamma_1) \oplus \dots \oplus X(\gamma_t)$ is rigid if and only if $M(\gamma_1) \oplus \dots \oplus M(\gamma_t)$ is rigid.*

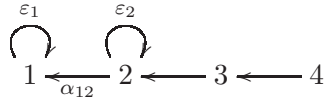
Given H , the rank vectors of the modules P_i^H are known [GLS1, Lemma 3.2]. For any H -preprojective positive root α , we have $\alpha = c_H^{-k}(\text{rank}(P_i^H))$ for uniquely determined $1 \leq i \leq n$ and $k \geq 0$. Thus the computation of the dimensions of Hom and Ext spaces for preprojective modules is purely combinatorial.

A tuple of indecomposable H -modules (M_1, \dots, M_t) is a *cycle* if there exists a non-zero non-isomorphism $M_i \rightarrow M_{i+1}$ for all $1 \leq i \leq t-1$ and $M_1 \cong M_t$. Such a cycle is a *strict cycle* if additionally $M_i \not\cong M_{i+1}$ for all $1 \leq i \leq t-1$.

As a consequence of Proposition 5.5 and the corresponding well known statement for preprojective T -modules, we get the following result.

Corollary 5.8. *There are no strict cycles consisting of preprojective H -modules.*

5.3. Example. Let $H = H(C, D, \Omega)$ be defined by the quiver



with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$ and $\varepsilon_1 \alpha_{12} = \alpha_{12} \varepsilon_2$. Thus H is of Dynkin type F_4 . Let $T = T(C, D, \Omega)$ be the corresponding hereditary algebra over some suitable ground field F . The Auslander-Reiten quiver of T is shown in Figure 1. The vertices of the quiver are the positive roots and stand for the preprojective T -modules $X(\alpha)$. (Since C is of Dynkin type, all indecomposable T -modules are preprojective.) For each non-projective $X(\alpha)$ there is a dashed arrow $\tau_T(X(\alpha)) \leftarrow -X(\alpha)$. Recall that we have $c_T(\underline{\dim}(X(\alpha))) = \underline{\dim}(\tau_T(X(\alpha)))$ and $c_H(\underline{\text{rank}}(M(\alpha))) = \underline{\text{rank}}(\tau_H(M(\alpha)))$. We can use now Figure 1 to compute for example the dimension of $\text{Hom}_H(M(2, 3, 4, 2), M(1, 2, 3, 2))$. We have $\tau_H^2(M(2, 3, 4, 2)) \cong M(1, 1, 0, 0) = P_2^H$ and $\tau_H^2(M(1, 2, 3, 2)) \cong M(1, 2, 2, 1)$. Thus

$$\begin{aligned} \dim \text{Hom}_H(M(2, 3, 4, 2), M(1, 2, 3, 2)) &= \dim \text{Hom}_H(P_2^H, M(1, 2, 2, 1)) \\ &= [M(1, 2, 2, 1) : S_2^H] \\ &= c_2(\underline{\text{rank}}(M(1, 2, 2, 1)))_2 \\ &= 2(1, 2, 2, 1)_2 = 4. \end{aligned}$$

5.4. Geometry of extension varieties. Let $H = H(C, D, \Omega)$. Let M and N be rigid locally free H -modules with $\text{Ext}_H^1(N, M) = 0$. Let $\mathbf{d} = (d_1, \dots, d_n) = \underline{\dim}(M) + \underline{\dim}(N)$

and $\mathbf{r} = (r_1, \dots, r_n)$ with $r_i = d_i/c_i$ for all i . Let $\mathcal{E}(M, N)$ be the set of all $E \in \text{rep}(H, \mathbf{d})$ such that there exists a short exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0.$$

It follows from [CBS, Theorem 1.3] that $\mathcal{E}(M, N)$ is irreducible and open in $\text{rep}(H, \mathbf{d})$. Furthermore, since $\text{rep}_{\text{l.f.}}(H)$ is closed under extensions, we know that $\mathcal{E}(M, N)$ is contained in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$. By [GLS2, Proposition 3.1] we know that $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ is irreducible and open in $\text{rep}(H, \mathbf{d})$. Thus $\mathcal{E}(M, N)$ is open and dense in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$.

Suppose now additionally, that $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ contains a rigid module R . Since $\mathcal{E}(M, N)$ is $G_{\mathbf{d}}$ -stable, open and dense in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ and since the $G_{\mathbf{d}}$ -orbit of a rigid module is also open, we get that $R \in \mathcal{E}(M, N)$. In other words, there exists a short exact sequence

$$0 \rightarrow N \rightarrow R \rightarrow M \rightarrow 0.$$

5.5. Construction of pseudo Auslander-Reiten sequences. As before, let $H = H(C, D, \Omega)$. Let α be an H -preprojective positive root such that α is not the rank vector of a projective H -module. The Auslander-Reiten sequence in $\text{mod}(T)$ ending in $X(\alpha)$ is of the form

$$0 \rightarrow \tau_T(X(\alpha)) \rightarrow \bigoplus_{i=1}^t X(\gamma_i)^{m_i} \rightarrow X(\alpha) \rightarrow 0$$

where $\gamma_1, \dots, \gamma_t$ are pairwise different H -preprojective roots and $m_i \geq 1$ for all i .

Theorem 5.9. *There is a short exact sequence*

$$0 \rightarrow \tau_H(M(\alpha)) \rightarrow \bigoplus_{i=1}^t M(\gamma_i)^{m_i} \rightarrow M(\alpha) \rightarrow 0$$

of H -modules.

Proof. We have $\text{Ext}_T^1(\tau_T(X(\alpha)), X(\alpha)) = 0$, and therefore $\text{Ext}_H^1(\tau_H(M(\alpha)), M(\alpha)) = 0$. From Section 5.4 we know that $\mathcal{E}(M(\alpha), \tau_H(M(\alpha)))$ is $G_{\mathbf{d}}$ -stable, open and dense in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$ where $\mathbf{d} = (d_1, \dots, d_n) = \underline{\dim}(M(\alpha)) + \underline{\dim}(\tau_H(M(\alpha)))$ and $\mathbf{r} = (r_1, \dots, r_n)$ with $r_i = d_i/c_i$ for all i . Since $\bigoplus_{i=1}^t X(\gamma_i)^{m_i}$ is a rigid T -module, we know that $R := \bigoplus_{i=1}^t M(\gamma_i)^{m_i}$ is a rigid locally free H -module with dimension vector \mathbf{d} . Thus the $G_{\mathbf{d}}$ -orbit of R is open and dense in $\text{rep}_{\text{l.f.}}(H, \mathbf{r})$. It follows that $R \in \mathcal{E}(M(\alpha), \tau_H(M(\alpha)))$. This finishes the proof. \square

The short exact sequence appearing in Theorem 5.9 is called a *pseudo Auslander-Reiten sequence*, or short a *pseudo AR-sequence*.

Let

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

be a pseudo AR-sequence of preprojective H -modules, and let M be an indecomposable τ -locally free H -module which is not isomorphic to Z . Let $h: M \rightarrow Z$ be a non-zero homomorphism. By [GLS1, Lemma 11.7 and Theorem 11.10] this implies that M is preprojective. By the Auslander-Reiten formula and since M is locally free, we get that $\text{Ext}_H^1(M, X) \cong D \text{Hom}_H(\tau_H^{-1}(X), M)$. By Proposition 5.8 and the fact that $\tau_H^{-1}(X) \cong Z$ we get $\text{Ext}_H^1(M, X) = 0$. Thus h factors through g . Similarly, if M is an indecomposable τ -locally free H -module which is not isomorphic to X , and if $h: X \rightarrow M$ is a non-zero homomorphism, then h factors through f . This shows that pseudo Auslander-Reiten sequences share at least some properties with Auslander-Reiten sequences. However, note

that for genuine Auslander-Reiten sequences, the above factorization properties hold for *every* indecomposable module M , not just for the τ -locally free ones.

6. CONSTRUCTION OF PRIMITIVE ELEMENTS

Let C be a Cartan matrix of Dynkin type, let D be the minimal symmetrizer of C , and let Ω be an orientation for C . Let $H = H(C, D, \Omega)$ and $\mathcal{M} = \mathcal{M}(H)$. Recall that $\mathcal{P}(\mathcal{M})$ denotes the Lie algebra of primitive elements in \mathcal{M} . Let $\Delta^+ = \Delta^+(C)$ be the set of positive roots of the Lie algebra $\mathfrak{g}(C)$. For $\gamma \in \Delta^+$ let $\mathcal{P}(\mathcal{M})_\gamma$ be the subspace of $\mathcal{P}(\mathcal{M})$ of elements of degree γ . The aim of Section 6 is to prove the following result, which implies that $\mathcal{P}(\mathcal{M})_\gamma$ is non-zero.

Theorem 6.1. *For each $\gamma \in \Delta^+$ there exists a primitive element $\theta_\gamma \in \mathcal{P}(\mathcal{M})_\gamma$ such that $\theta_\gamma(M(\gamma)) = 1$.*

We prove Theorem 6.1 for each non-symmetric C of Dynkin type by explicitly constructing in Sections 6.5, 6.6, 6.7, 6.8 the elements θ_γ . For symmetric C of Dynkin type, H is a path algebra of finite type and for each positive root γ there is a unique indecomposable H -module of dimension vector γ . Therefore our claim follows from Schofield's theorem which shows that \mathfrak{n} is identified with the primitive elements of \mathcal{M} .

6.1. Admissible triples. A triple (α, β, γ) of positive roots is an *H-admissible triple* if the following hold:

- (c1) $\text{Hom}_H(M(\alpha), M(\beta)) = 0$ and $\text{Hom}_H(M(\beta), M(\alpha)) = 0$;
- (c2) There exists a short exact sequence

$$0 \rightarrow M(\alpha) \xrightarrow{f} M(\gamma) \xrightarrow{g} M(\beta) \rightarrow 0;$$

- (c3) For each indecomposable locally free submodule U of $M(\gamma)$ with $\underline{\text{rank}}(U) = \alpha$ we have $U \cong M(\alpha)$, or for each indecomposable locally free factor module V of $M(\gamma)$ with $\underline{\text{rank}}(V) = \beta$ we have $V \cong M(\beta)$.

Lemma 6.2. *Assume that (α, β, γ) is a triple of positive roots satisfying (c2). Then the following hold:*

- (i) $M(\gamma)$ does not have any locally free submodule U with $\underline{\text{rank}}(U) = \beta$;
- (ii) $M(\gamma)$ does not have any locally free factor module V with $\underline{\text{rank}}(V) = \alpha$.

Proof. Assume U is a submodule of $M(\gamma)$ with rank vector β . We have

$$\langle \beta, \beta \rangle_H = \dim \text{Hom}_H(M(\beta), U) - \dim \text{Ext}_H^1(M(\beta), U) > 0.$$

Thus we get $\text{Hom}_H(M(\beta), U) \neq 0$. But this implies $\text{Hom}_H(M(\beta), M(\gamma)) \neq 0$, a contradiction, since there are no strict cycles consisting of preprojective H -modules, see Proposition 5.8. Part (ii) is proved dually. \square

Lemma 6.3. *Assume that (α, β, γ) is a triple of positive roots satisfying (c1) and (c2). Then the following hold:*

- (i) $M(\gamma)$ has only one submodule isomorphic to $M(\alpha)$, namely $\text{Im}(f)$;
- (ii) For any locally free submodule U of $M(\gamma)$ with $\underline{\text{rank}}(U) = \alpha$ and $U \not\cong M(\alpha)$ we have

$$\text{Hom}_H(U, M(\beta)) \neq 0;$$

- (iii) $M(\gamma)$ has only one factor module isomorphic to $M(\beta)$, namely $M(\gamma)/\text{Im}(f)$;

- (iv) For any locally free factor module V of $M(\gamma)$ with $\underline{\text{rank}}(V) = \beta$ and $V \not\cong M(\beta)$ we have

$$\text{Hom}_H(M(\alpha), V) \neq 0.$$

Proof. Let $f': U \rightarrow M(\gamma)$ be a monomorphism where $\underline{\text{rank}}(U) = \alpha$. Suppose that $g \circ f' = 0$. Then for dimension reasons we get $\text{Im}(f) = \text{Im}(f')$. If $g \circ f' \neq 0$, then $U \not\cong M(\alpha)$ since $\text{Hom}_H(M(\alpha), M(\beta)) = 0$. This yields (i) and (ii). Parts (iii) and (iv) are proved dually. \square

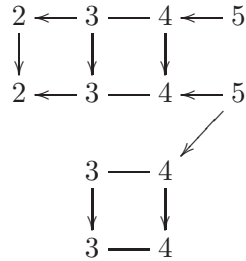
Proposition 6.4. *Let (α, β, γ) be an H -admissible triple. Let $\rho, \sigma \in \mathcal{P}(\mathcal{M})$ such that $\rho(M(\alpha)) = 1$ and $\sigma(M(\beta)) = 1$. Then $[\rho, \sigma] \in \mathcal{P}(\mathcal{M})$ satisfies $[\rho, \sigma](M(\gamma)) = 1$.*

Proof. We have $[\rho, \sigma] = \rho * \sigma - \sigma * \rho$. Now $M(\gamma)$ has exactly one indecomposable submodule U with rank vector α and we have $U \cong M(\alpha)$ and $M(\gamma)/U \cong M(\beta)$, or $M(\gamma)$ has exactly one indecomposable factor module $V = M(\gamma)/U$ with rank vector β and we have $V \cong M(\beta)$ and $U \cong M(\alpha)$. In both cases, we get $(\rho * \sigma)(M(\gamma)) = 1$. Furthermore, $M(\gamma)$ has no submodule with rank vector β . Thus we have $(\sigma * \rho)(M(\gamma)) = 0$. This finishes the proof. \square

6.2. Convention for displaying modules. Let $H = H(C, D, \Omega)$ with C of Dynkin type. Thus the quiver $Q = Q(C, \Omega)$ has no multiple arrows. In the following sections we will construct explicitly some H -modules M by giving a \mathbb{C} -basis B_i of M_i for each vertex i of Q together with the action of the arrows of H on the basis $B := B_1 \cup \dots \cup B_n$ of M . This information is displayed in a diagram. The vertices of the diagram with label i correspond to the elements in B_i , and the edges in the diagram show how the arrows of Q act on B . For example, let

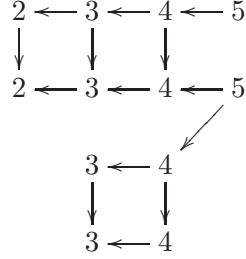
$$C = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

be a Cartan matrix of type B_5 . The minimal symmetrizer of C is $D = \text{diag}(2, 2, 2, 2, 1)$. Let Ω be an orientation of C with $\{(2, 3), (4, 5)\} \subset \Omega$. The diagram



defines a locally free H -module M with $\underline{\text{rank}}(M) = (0, 1, 2, 2, 2)$. Each vertex i of the diagram stands for a basis vector of M_i . An oriented edge like $2 \leftarrow 3$ means that α_{23} sends the basis vector labeled by 3 to the one labeled by 2. A non-oriented edge like $3 \text{ --- } 4$ means that α_{34} sends the basis vector labeled by 4 to the one labeled by 3 provided $(3, 4) \in \Omega$, or that α_{43} sends the basis vector labeled by 3 to the one labeled by 4 in case $(4, 3) \in \Omega$. The two arrows starting in the lower 5 mean that α_{45} sends the basis vector labeled by the lower 5 to the sum of the basis vectors labeled by the two 4's in the middle.

6.3. How do we know that we constructed the correct module? Let C and D be as in the example in Section 6.2, and set $\Omega = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$. Let $H = H(C, D, \Omega)$. Let M be the H -module defined by the diagram



Clearly, M is a locally free H -module with rank vector $\mathbf{r} := (0, 1, 2, 2, 2)$. (Just check that all defining relations for H are satisfied and that the H_i -module M_i is free for all i . This is straightforward.) Furthermore, we claim that M is isomorphic to the indecomposable preprojective H -module $M(\gamma)$ with $\gamma = \mathbf{r}$. To prove this, we need to show that M is rigid, compare [GLS1, Theorem 1.2]. Let $\mathbf{d} = \underline{\dim}(M) = (0, 2, 4, 4, 2)$. Recall that $q_H(\mathbf{r}) = \dim \text{End}_H(M) - \dim \text{Ext}_H^1(M, M)$. Thus M is rigid if and only if $\dim \text{End}_H(M) = q_H(\mathbf{r})$. The proof of this equality is done by an explicit calculation which we now carry out for the above example. In all other examples below, this is done similarly and is left to the reader.

We have $M = (M_i, M(\alpha_{ij}), M(\varepsilon_i))$ with $(i, j) \in \Omega$ and $1 \leq i \leq 5$. We have $M_1 = 0$, $M_2 = K^2$, $M_3 = K^4$, $M_4 = K^4$ and $M_5 = K^2$. Numbering the basis vectors in each column of the above diagram from bottom to top we get

$$M(\varepsilon_1) = 0, \quad M(\varepsilon_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M(\varepsilon_3) = M(\varepsilon_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M(\alpha_{12}) = 0,$$

$$M(\alpha_{23}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M(\alpha_{34}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M(\alpha_{45}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

An endomorphism of M is given by a tuple $F = (F_1, F_2, F_3, F_4, F_5)$ with $F_1 = 0$, $F_2, F_5 \in M_2(K)$ and $F_3, F_4 \in M_4(K)$ such that the following relations hold:

$$F_i M(\varepsilon_i) = M(\varepsilon_i) F_i \quad \text{for } i = 2, 3, 4, \quad (6.1)$$

$$F_2 M(\alpha_{23}) = M(\alpha_{23}) F_3, \quad (6.2)$$

$$F_3 M(\alpha_{34}) = M(\alpha_{34}) F_4, \quad (6.3)$$

$$F_4 M(\alpha_{45}) = M(\alpha_{45}) F_5. \quad (6.4)$$

Equation (6.3) implies that $F_3 = F_4$. From (6.1) we get that F_2, \dots, F_5 are of the form

$$F_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}, \quad F_3 = F_4 = \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ 0 & a_3 & 0 & c_3 \\ e_3 & f_3 & g_3 & h_3 \\ 0 & e_3 & 0 & g_3 \end{pmatrix}, \quad F_5 = \begin{pmatrix} a_5 & b_5 \\ c_5 & d_5 \end{pmatrix}.$$

with $a_i, b_i, c_j, d_j, e_3, f_3, g_3, h_3 \in K$ for $i = 2, 3, 5$ and $j = 3, 5$. Now (6.2) yields $e_3 = f_3 = 0$, $a_2 = g_3$ and $b_2 = h_3$. Equation (6.4) implies that $c_3 = b_2 = -b_3 = b_5$, $d_3 = c_5 = 0$ and

$a_2 = a_3 = a_5 = d_5$. Combining all equations we get

$$F_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}, \quad F_3 = F_4 = \begin{pmatrix} a_2 & -b_2 & b_2 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \quad F_5 = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}.$$

This shows that $\dim \operatorname{End}_H(M) = 2 = q_H(\mathbf{r})$. Thus M is rigid, and therefore $M \cong M(\gamma)$.

6.4. Filtrations. Let $\Phi := (\beta_1, \dots, \beta_t)$ be a sequence of positive roots. We say that an H -module M has a *filtration of type Φ* if there is a chain

$$0 = M_0 \subset M_1 \subset \dots \subset M_{t-1} \subset M_t = M$$

of submodules M_i of M such that M_i/M_{i-1} is locally free with $\underline{\operatorname{rank}}(M_i/M_{i-1}) = \beta_i$ for all $1 \leq i \leq t$.

Suppose that $\theta_{\beta_i} \in \mathcal{M}_{\beta_i}$ for $1 \leq i \leq t$. If $\theta_{\beta_1} \dots \theta_{\beta_t}(M) \neq 0$ for some H -module M , then M needs to have a filtration of type $(\beta_1, \dots, \beta_t)$. This obvious fact will be used at various places in the following sections.

6.5. Type B_n . Let

$$C = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

be a Cartan matrix of type B_n . The minimal symmetrizer of C is $D = \operatorname{diag}(2, \dots, 2, 1)$. Let Ω be an orientation of C such that $(n-1, n) \in \Omega$. (The case $(n, n-1) \in \Omega$ is treated dually.) Set $H = H(C, D, \Omega)$. We construct now explicitly for each $\gamma \in \Delta^+$ the indecomposable preprojective H -module $M(\gamma)$ with rank vector γ and the corresponding primitive element θ_γ .

6.5.1. For $1 \leq k \leq s \leq n-1$ and $\gamma = \sum_{i=k}^s \alpha_i$, we have

$$M(\gamma): \begin{array}{ccccccc} k & \text{---} & k+1 & \text{---} & \dots & \text{---} & s \\ \downarrow & & \downarrow & & & & \downarrow \\ k & \text{---} & k+1 & \text{---} & \dots & \text{---} & s \end{array}$$

If $k = s$, set $\theta_\gamma := \theta_k$. Thus let $k < s$. For $(k+1, k) \in \Omega$, set $\alpha = \sum_{i=k+1}^s \alpha_i$ and $\beta = \gamma - \alpha = \alpha_k$, and for $(k, k+1) \in \Omega$, let $\alpha = \alpha_k$ and $\beta = \gamma - \alpha = \sum_{i=k+1}^s \alpha_i$. In both cases it is now easy to check that (α, β, γ) is an H -admissible triple. Set $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

6.5.2. For $1 \leq k \leq n-1$ and $\gamma = \sum_{i=k}^n \alpha_i$, we have

$$M(\gamma): \begin{array}{ccccccc} k & \text{---} & \dots & \text{---} & n-1 & \longleftarrow & n \\ \downarrow & & & & \downarrow & & \\ k & \text{---} & \dots & \text{---} & n-1 & & \end{array}$$

Take $\alpha = \sum_{i=k}^{n-1} \alpha_i$ and $\beta = \gamma - \alpha = \alpha_n$. It is easy to check that (α, β, γ) is an H -admissible triple. Define $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

6.5.3. For $1 \leq k \leq n-1$ and $\gamma = (\sum_{i=k}^{n-1} \alpha_i) + 2\alpha_n$, we have

$$M(\gamma): \begin{array}{ccccccc} k & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \\ \downarrow & & & & \downarrow & & \\ k & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \end{array}$$

Take $\alpha = \sum_{i=k}^{n-1} \alpha_i$ and $\beta = \alpha_n$, and define $\theta_\gamma := 1/2[[\theta_\alpha, \theta_\beta], \theta_\beta]$. It is straightforward to check that $M(\gamma)$ does not have any filtrations of type (β, α, β) or (β, β, α) . We get $\theta_\gamma(M(\gamma)) = 1$.

6.5.4. For $1 \leq k \leq s \leq n-2$ and $\gamma = \sum_{i=k}^s \alpha_i + \sum_{j=s+1}^n 2\alpha_j$, we have

$$M(\gamma): \begin{array}{ccccccccccc} k & \text{---} & \cdots & \text{---} & s & \longleftarrow & s+1 & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \\ k & \text{---} & \cdots & \text{---} & s & \longleftarrow & s+1 & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \\ & & & & & & & & & & \swarrow & & \\ & & & & & & & & & & s+1 & \text{---} & \cdots & \text{---} & n-1 \\ & & & & & & & & & & \downarrow & & & & \downarrow \\ & & & & & & & & & & s+1 & \text{---} & \cdots & \text{---} & n-1 \end{array}$$

if $(s, s+1) \in \Omega$, and

$$M(\gamma): \begin{array}{ccccccccccc} & & & & & & s+1 & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \\ & & & & & & \downarrow & & & & \downarrow & & \\ & & & & & & s+1 & \text{---} & \cdots & \text{---} & n-1 & \longleftarrow & n \\ & & & & & & & & & & \swarrow & & \\ k & \text{---} & \cdots & \text{---} & s & \longrightarrow & s+1 & \text{---} & \cdots & \text{---} & n-1 \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ k & \text{---} & \cdots & \text{---} & s & \longrightarrow & s+1 & \text{---} & \cdots & \text{---} & n-1 \end{array}$$

if $(s+1, s) \in \Omega$. Let $t \geq s+1$ be maximal such that there is a path p in Q with $s(p) = s+1$ and $t(p) = t$. For $(s, s+1) \in \Omega$, set $\alpha = \sum_{i=s+1}^t \alpha_i$ and $\beta = \gamma - \alpha$, and for $(s+1, s) \in \Omega$, set $\alpha = \sum_{i=k}^t \alpha_i$ and $\beta = \gamma - \alpha$. In both cases it is now easy to check that (α, β, γ) is an H -admissible triple. Set $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

6.6. **Type C_n .** Let

$$C = \begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

be a Cartan matrix of type C_n . The minimal symmetrizer of C is $D = \text{diag}(2, 1, \dots, 1)$. Let Ω be an orientation of C . Let $H = H(C, D, \Omega)$. We construct now explicitly for each $\gamma \in \Delta^+$ the indecomposable preprojective H -module $M(\gamma)$ with rank vector γ and the corresponding primitive element θ_γ .

6.6.1. For $2 \leq k \leq s \leq n$ and $\gamma = \sum_{i=k}^s \alpha_i$, we have

$$M(\gamma): \quad k \text{ --- } k+1 \text{ --- } \cdots \text{ --- } s$$

For $(k+1, k) \in \Omega$, let $\alpha = \sum_{i=k+1}^s \alpha_i$ and $\beta = \gamma - \alpha = \alpha_k$, and for $(k, k+1) \in \Omega$, let $\alpha = \alpha_k$ and $\beta = \gamma - \alpha = \sum_{i=k+1}^s \alpha_i$. In both cases it is easy to check that (α, β, γ) is an H -admissible triple. Set $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

6.6.2. For $1 \leq k \leq n$ and $\gamma = \alpha_1 + \sum_{i=2}^k 2\alpha_i$, we have

$$\begin{array}{c} M(\gamma): \quad 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \\ \downarrow \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \end{array}$$

For $k=1$ let $\theta_\gamma := \theta_1$. Assume that $k > 1$. If $(k-1, k) \in \Omega$, then set $\alpha = \alpha_1 + \sum_{i=2}^{k-1} 2\alpha_i$ and $\beta = \alpha_k$, and let $\theta_\gamma := 1/2[[\theta_\alpha, \theta_\beta], \theta_\beta]$. It is straightforward to check that $M(\gamma)$ does not have any filtrations of type (β, α, β) or (β, β, α) . We get $\theta_\gamma(M(\gamma)) = 1$. If $(k, k-1) \in \Omega$, then set $\alpha = \alpha_k$ and $\beta = \alpha_1 + \sum_{i=2}^{k-1} 2\alpha_i$, and let $\theta_\gamma := 1/2[\theta_\alpha, [\theta_\alpha, \theta_\beta]]$. It is straightforward to check that $M(\gamma)$ does not have any filtrations of type (α, β, α) or (β, α, α) . We get $\theta_\gamma(M(\gamma)) = 1$.

6.6.3. For $1 \leq k < s \leq n$ and $\gamma = \alpha_1 + \sum_{i=2}^k 2\alpha_i + \sum_{j=k+1}^s \alpha_j$, we have

$$\begin{array}{c} M(\gamma): \quad 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \\ \downarrow \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \longrightarrow k+1 \text{ --- } \cdots \text{ --- } s \end{array}$$

if $(k+1, k) \in \Omega$, and

$$\begin{array}{c} M(\gamma): \quad 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \longleftarrow k+1 \text{ --- } \cdots \text{ --- } s \\ \downarrow \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } k \end{array}$$

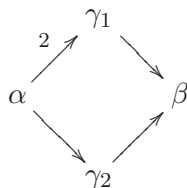
if $(k, k+1) \in \Omega$. For $(k+1, k) \in \Omega$, let $\alpha = \sum_{i=k+1}^s \alpha_i$ and $\beta = \gamma - \alpha = \alpha_1 + \sum_{i=2}^k 2\alpha_i$, and for $(k, k+1) \in \Omega$, set $\alpha = \alpha_1 + \sum_{i=2}^k 2\alpha_i$ and $\beta = \gamma - \alpha = \sum_{j=k+1}^s \alpha_j$. In both cases it is easy to check that (α, β, γ) is an H -admissible triple. Let $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

6.7. **Type F_4 .** Let

$$C = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

be a Cartan matrix of type F_4 . The minimal symmetrizer of C is $D = \text{diag}(2, 2, 1, 1)$. Recall that a root $\gamma \in \Delta^+$ is *sincere* provided γ is of the form $\gamma = \sum_{i=1}^4 m_i \alpha_i$ with $m_i \geq 1$ for all i . For the non-sincere roots $\gamma \in \Delta^+$, we assume that θ_γ is already constructed (using the Dynkin types B_3 and C_3). There are 8 orientations for type F_4 . In the following Figures 1, 2, 3, 4 we display the Auslander-Reiten quivers of the hereditary tensor algebras

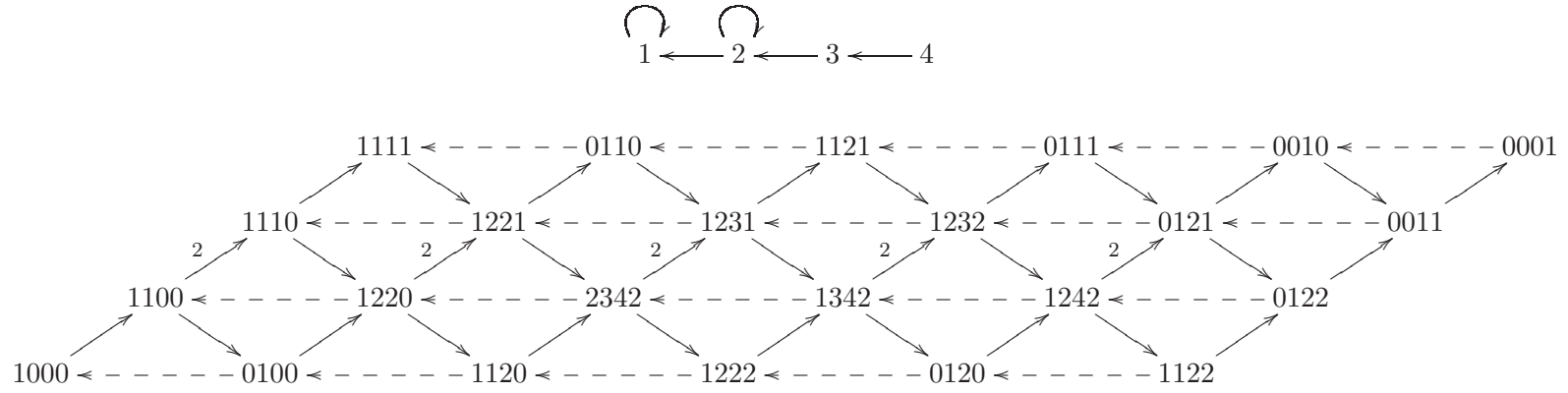
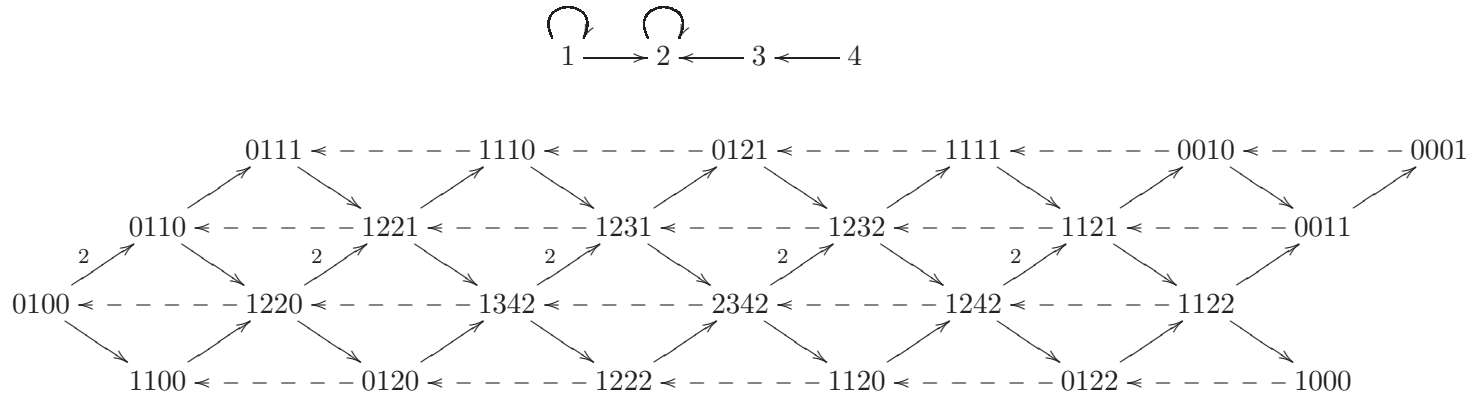
$T(C, D, \Omega)$ of type F_4 with 4 of these 8 orientations Ω . The other 4 orientations are treated dually. The subgraphs of the form

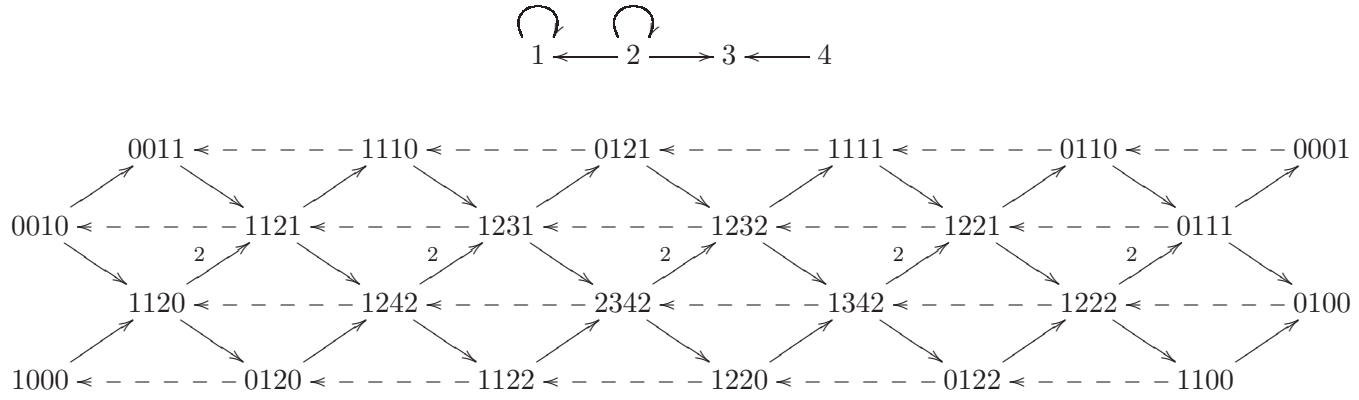
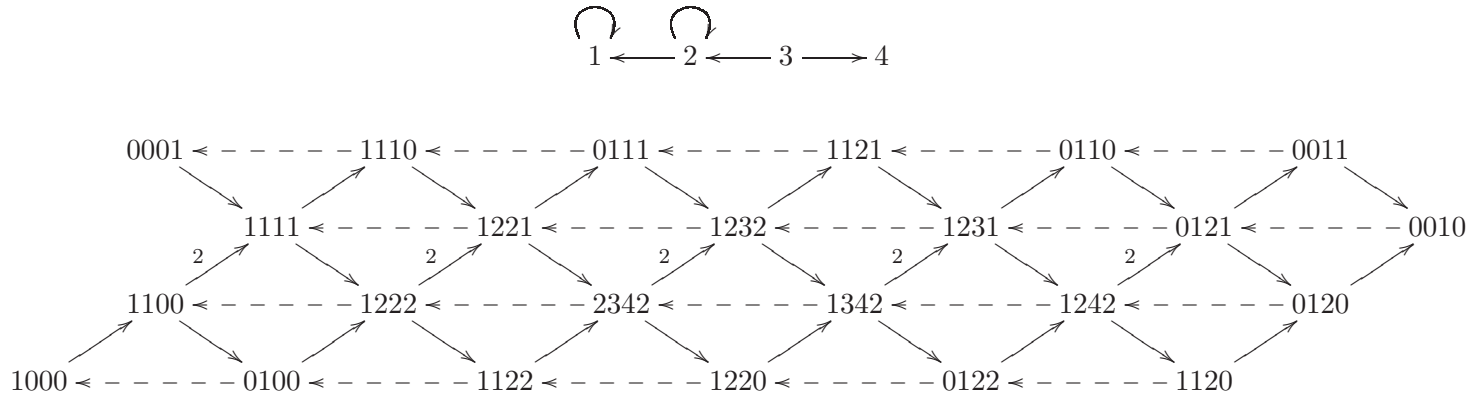


stand for Auslander-Reiten sequences

$$0 \rightarrow X(\alpha) \rightarrow X(\gamma_1)^2 \oplus X(\gamma_2) \rightarrow X(\beta) \rightarrow 0.$$

For each orientation we have 10 sincere roots $\gamma \in \Delta^+$. Each of these will be treated separately. So in total we need to consider 40 different cases. In each case, we construct for a given γ an H -admissible triple (α, β, γ) . For all pairs (α, β) appearing in our constructions, one uses the techniques explained in Section 5.2 to check that $\text{Hom}_H(M(\alpha), M(\beta)) = 0$ and $\text{Hom}_H(M(\beta), M(\alpha)) = 0$. This is left to the reader.

FIGURE 1. Type F_4 , Case 1FIGURE 2. Type F_4 , Case 2


 FIGURE 3. Type F_4 , Case 3

 FIGURE 4. Type F_4 , Case 4

If not mentioned otherwise, in all of the following cases we define $\theta_\gamma := [\theta_\alpha, \theta_\beta]$.

In the following sections, the 40 cases considered are labeled by $x.1, \dots, x.10$ with $x = 1, 2, 3, 4$. The x indicates which of the four orientations Ω displayed in Figures 1,2,3,4 we are working with. With respect to the diagrams displayed in Figures 1,2,3,4, we go through the cases row-wise. Throughout, the logic and results of Section 6.1 will be used freely without reference.

6.7.1. *Case 1.1.* Let $\gamma = (1, 1, 1, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 0), (0, 0, 0, 1))$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \leftarrow 2 \leftarrow 3 & M(\gamma): 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\ & 1 \leftarrow 2 & 1 \leftarrow 2 \end{array}$$

There is an obvious short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) It is easily checked that (α, β, γ) is an H -admissible triple.

6.7.2. *Case 1.2.* Let $\gamma = (1, 1, 2, 1)$ and $(\alpha, \beta) = ((1, 1, 2, 0), (0, 0, 0, 1))$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \leftarrow 2 \leftarrow 3 & M(\gamma): 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 & 1 \leftarrow 2 \leftarrow 3 \end{array}$$

There is an obvious short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) Obviously, (α, β, γ) is an H -admissible triple.

6.7.3. *Case 1.3.* Let $\gamma = (1, 2, 2, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 1), (0, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{c} M(\beta): 2 \leftarrow 3 \\ \downarrow \\ 2 \end{array}$$

The only other locally free indecomposable H -module with rank vector $(0, 1, 1, 0)$ is

$$\begin{array}{c} V(\beta): 2 \\ \downarrow \\ 2 \leftarrow 3 \end{array}$$

We have $\text{Hom}_H(V(\beta), E_2) \neq 0$. Using Figure 1 we get $\text{Hom}_H(M(\gamma), E_2) = 0$. Thus $V(\beta)$ cannot be a factor module of $M(\gamma)$. It follows that (α, β, γ) is an H -admissible triple.

6.7.4. *Case 1.4.* Let $\gamma = (1, 2, 3, 1)$ and $(\alpha, \beta) = ((0, 1, 1, 0), (1, 1, 2, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 2 \leftarrow 3 & M(\beta): 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow & \downarrow \quad \downarrow \\ & 2 & 1 \leftarrow 2 \leftarrow 3 \end{array}$$

The only other locally free indecomposable H -module with rank vector α is

$$\begin{array}{c} U(\alpha): 2 \\ \downarrow \\ 2 \leftarrow 3 \end{array}$$

We have $\text{Hom}_H(U(\alpha), M(\beta)) = 0$, thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.5. *Case 1.5.* Let $\gamma = (1, 2, 3, 2)$ and $(\alpha, \beta) = ((1, 1, 2, 1), (0, 1, 1, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$M(\beta): \begin{array}{c} 2 \leftarrow 3 \leftarrow 4 \\ \downarrow \\ 2 \end{array}$$

The only other locally free indecomposable H -module with rank vector β is

$$V(\beta): \begin{array}{c} 2 \\ \downarrow \\ 2 \leftarrow 3 \leftarrow 4 \end{array}$$

We have $\text{Hom}_H(V(\beta), E_2) \neq 0$. Using Figure 1 we see that $\text{Hom}_H(M(\gamma), E_2) = 0$. Thus $V(\beta)$ cannot be a factor module of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.6. *Case 1.6.* Let $\gamma = (2, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 1, 2, 0), (1, 2, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$M(\alpha): \begin{array}{c} 1 \leftarrow 2 \leftarrow 3 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \leftarrow 3 \end{array} \quad M(0, 1, 1, 0): \begin{array}{c} 2 \leftarrow 3 \\ \downarrow \\ 2 \end{array}$$

The only other locally free indecomposable H -module with rank vector α is

$$U(\alpha): \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \leftarrow 3 \\ \downarrow \\ 2 \leftarrow 3 \end{array}$$

We have $\text{Hom}_H(M(0, 1, 1, 0), U(\alpha)) \neq 0$. On the other hand, using Figure 1 we get $\text{Hom}_H(M(0, 1, 1, 0), M(\gamma)) = 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.7. *Case 1.7.* Let $\gamma = (1, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 2, 2, 2), (0, 1, 2, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.8. *Case 1.8.* Let $\gamma = (1, 2, 4, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 0), (1, 1, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.9. *Case 1.9.* Let $\gamma = (1, 2, 2, 2)$ and $(\alpha, \beta) = ((0, 1, 0, 0), (1, 1, 2, 2))$. We have

$$M(\gamma): \begin{array}{c} 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ \swarrow \\ 2 \\ \downarrow \\ 2 \end{array} \quad M(\beta): \begin{array}{c} 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) We also see that there is only one locally free indecomposable H -module with rank vector α , namely $M(\alpha)$. Thus (α, β, γ) is an H -admissible triple.

6.7.10. *Case 1.10.* Let $\gamma = (1, 1, 2, 2)$ and $(\alpha, \beta) = ((1, 0, 0, 0), (0, 1, 2, 2))$. We have

$$\begin{array}{ccc} M(\gamma): & 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 & M(\beta): & 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 & & 2 \leftarrow 3 \leftarrow 4 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) There is only one locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.11. *Case 2.1.* $\gamma = (1, 1, 1, 1)$, $(\alpha, \beta) = ((1, 1, 1, 0), (0, 0, 0, 1))$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \rightarrow 2 \leftarrow 3 & M(\gamma): & 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ & 1 \rightarrow 2 & & 1 \rightarrow 2 \end{array}$$

There is a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) Obviously, (α, β, γ) is an H -admissible triple.

6.7.12. *Case 2.2.* Let $\gamma = (1, 2, 2, 1)$ and $(\alpha, \beta) = ((0, 1, 1, 1), (1, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\beta): & 1 \rightarrow 2 \leftarrow 3 & M(1, 1, 0, 0): & 1 \rightarrow 2 & M(0, 1, 2, 0): & 2 \leftarrow 3 \\ & \downarrow \quad \downarrow & & \downarrow \quad \downarrow & & \downarrow \\ & 1 \rightarrow 2 & & 1 \rightarrow 2 & & 2 \leftarrow 3 \end{array}$$

The only one other locally free indecomposable H -modules with rank vector β are

$$\begin{array}{ccc} V_1(\beta): & 1 \rightarrow 2 & V_2(\beta): & 2 \leftarrow 3 & V_3(\beta): & 2 \\ & \downarrow \quad \downarrow & & \downarrow & & \downarrow \\ & 1 \rightarrow 2 \leftarrow 3 & & 1 \rightarrow 2 & & 1 \rightarrow 2 \leftarrow 3 \\ & & & \downarrow & & \downarrow \\ & & & 1 & & 1 \end{array}$$

By Figure 2 we have $\text{Hom}_H(M(\gamma), M(0, 1, 2, 0)) = 0$. For $i = 2, 3$ we clearly have $\text{Hom}_H(V_i(\beta), M(0, 1, 2, 0)) \neq 0$. Thus $V_i(\beta)$ cannot be a factor module of $M(\gamma)$ for $i = 2, 3$. Figure 2 shows that $\text{Hom}_H(M(\gamma), M(1, 1, 0, 0)) = 0$. On the other hand, we have $\text{Hom}_H(V_1(\beta), M(1, 1, 0, 0)) \neq 0$. Thus $V_1(\beta)$ cannot be a factor module of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.13. *Case 2.3.* Let $\gamma = (1, 2, 3, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 0), (0, 1, 2, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \rightarrow 2 \leftarrow 3 & M(\beta): & 2 \leftarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \rightarrow 2 & & 2 \leftarrow 3 \end{array}$$

The only other locally free indecomposable H -modules with rank vector α are

$$\begin{array}{ccc}
 U_1(\alpha): & \begin{array}{c} 1 \longrightarrow 2 \\ \downarrow \quad \downarrow \\ 1 \longrightarrow 2 \longleftarrow 3 \end{array} & U_2(\alpha): \quad \begin{array}{c} 2 \longleftarrow 3 \\ \downarrow \\ 1 \longrightarrow 2 \\ \downarrow \\ 1 \end{array} & U_3(\alpha): \quad \begin{array}{c} 2 \\ \downarrow \\ 1 \longrightarrow 2 \longleftarrow 3 \\ \downarrow \\ 1 \end{array}
 \end{array}$$

For $i = 2, 3$ we have $\text{Hom}_H(E_1, U_i(\alpha)) \neq 0$, and from Figure 2 we get $\text{Hom}_H(E_1, M(\gamma)) = 0$. Thus for $i = 2, 3$, $U_i(\alpha)$ cannot be a submodule of $M(\gamma)$. We obviously have $\text{Hom}_H(U_1(\alpha), M(\beta)) = 0$. This implies that $U_1(\alpha)$ cannot be a submodule of $M(\gamma)$. It follows that (α, β, γ) is an H -admissible triple.

6.7.14. *Case 2.4.* Let $\gamma = (1, 2, 3, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 1), (1, 1, 1, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{c} 2 \longleftarrow 3 \longleftarrow 4 \\ \downarrow \\ 2 \longleftarrow 3 \end{array} & M(\beta): \quad \begin{array}{c} 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \\ \downarrow \quad \downarrow \\ 1 \longrightarrow 2 \end{array}
 \end{array}$$

The only other locally free indecomposable H -module with rank vector α is

$$\begin{array}{c}
 U(\alpha): \quad 2 \longleftarrow 3 \\
 \downarrow \\
 2 \longleftarrow 3 \longleftarrow 4
 \end{array}$$

We have $\text{Hom}_H(U(\alpha), M(\beta)) = 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. It follows that (α, β, γ) is an H -admissible triple.

6.7.15. *Case 2.5.* Let $\gamma = (1, 1, 2, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 1), (0, 0, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.16. *Case 2.6.* Let $\gamma = (1, 3, 4, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 0), (1, 2, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. There is only one locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.17. *Case 2.7.* Let $\gamma = (2, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 2, 2, 2), (1, 1, 2, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc}
 M(\beta): & \begin{array}{c} 1 \longrightarrow 2 \longleftarrow 3 \\ \downarrow \quad \downarrow \\ 1 \longrightarrow 2 \longleftarrow 3 \end{array} & M(0, 1, 2, 0): \quad \begin{array}{c} 2 \longleftarrow 3 \\ \downarrow \\ 2 \longleftarrow 3 \end{array}
 \end{array}$$

There is only one other locally free indecomposable H -module with rank vector β , namely

$$\begin{array}{c}
 V(\beta): \quad \begin{array}{c} 2 \longleftarrow 3 \\ \downarrow \\ 1 \longrightarrow 2 \longleftarrow 3 \\ \downarrow \\ 1 \end{array}
 \end{array}$$

Using Figure 2 we get $\text{Hom}_H(M(\gamma), M(0, 1, 2, 0)) = 0$. Furthermore, we obviously have $\text{Hom}_H(V(\beta), M(0, 1, 2, 0)) \neq 0$. Thus $V(\beta)$ cannot be a factor module of $M(\gamma)$. It follows that (α, β, γ) is an H -admissible triple.

6.7.18. *Case 2.8.* Let $\gamma = (1, 2, 4, 2)$ and $(\alpha, \beta) = ((1, 1, 2, 0), (0, 1, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.19. *Case 2.9.* Let $\gamma = (1, 1, 2, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 2), (1, 0, 0, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.20. *Case 2.10.* Let $\gamma = (1, 2, 2, 2)$ and $(\alpha, \beta) = ((1, 1, 0, 0), (0, 1, 2, 2))$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ & 1 & \longrightarrow & 2 \end{array} & M(\gamma): & \begin{array}{ccc} 2 & \longleftarrow & 3 \longleftarrow & 4 \\ \downarrow & & \downarrow & \\ 2 & \longleftarrow & 3 \longleftarrow & 4 \end{array} & M(\beta): & \begin{array}{ccc} 2 & \longleftarrow & 3 \longleftarrow & 4 \\ \downarrow & & \downarrow & \\ 2 & \longleftarrow & 3 \longleftarrow & 4 \end{array} \\
 & & & & & \begin{array}{ccc} & & \swarrow \\ & & \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ & 1 & \longrightarrow & 2 \end{array}
 \end{array}
 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.21. *Case 3.1.* $\gamma = (1, 1, 1, 1)$, $(\alpha, \beta) = ((1, 1, 1, 0), (0, 0, 0, 1))$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{ccc} 1 & \longleftarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longleftarrow & 2 \longrightarrow & 3 \end{array} & M(\gamma): & \begin{array}{ccc} 1 & \longleftarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longleftarrow & 2 \longrightarrow & 3 \longleftarrow & 4 \end{array}
 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.22. *Case 3.2.* Let $\gamma = (1, 1, 2, 1)$ and $(\alpha, \beta) = ((0, 0, 1, 1), (1, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.23. *Case 3.3.* Let $\gamma = (1, 2, 3, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 0), (0, 1, 2, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{ccc} 1 & \longleftarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longleftarrow & 2 \longrightarrow & 3 \end{array} & M(\beta): & \begin{array}{ccc} 2 & \longrightarrow & 3 \longleftarrow & 4 \\ \downarrow & & \downarrow & \\ 2 & \longrightarrow & 3 \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 M(1, 1, 0, 0): & \begin{array}{ccc} 1 & \longleftarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longleftarrow & 2 \end{array} & M(0, 1, 1, 0): & \begin{array}{ccc} 2 \\ \downarrow \\ 2 \longrightarrow & 3 \end{array}
 \end{array}$$

There are only three more locally free indecomposable H -modules with rank vector α , namely

$$\begin{array}{ccc}
 U_1(\alpha): & \begin{array}{c} 1 \leftarrow 2 \rightarrow 3 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \end{array} & U_2(\alpha): \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \rightarrow 3 \\ \downarrow \\ 2 \end{array} & U_3(\alpha): \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \\ \downarrow \\ 2 \rightarrow 3 \end{array}
 \end{array}$$

For $i = 1, 2$ we have $\text{Hom}_H(M(1, 1, 0, 0), U_i(\alpha)) \neq 0$, and from Figure 3 we know that $\text{Hom}_H(M(1, 1, 0, 0), M(\gamma)) = 0$. Thus for $i = 1, 2$, $U_i(\alpha)$ cannot be a submodule of $M(\gamma)$. One easily checks that $\text{Hom}_H(U_3(\alpha), M(\beta)) = 0$. Thus $U_3(\alpha)$ cannot be a submodule of $M(\gamma)$. It follows that (α, β, γ) is an H -admissible triple.

6.7.24. *Case 3.4.* Let $\gamma = (1, 2, 3, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 1), (1, 1, 1, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{c} 2 \rightarrow 3 \leftarrow 4 \\ \downarrow \\ 2 \rightarrow 3 \end{array} & M(\beta): \begin{array}{c} 1 \leftarrow 2 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \end{array}
 \end{array}$$

The only other locally free indecomposable H -module with rank vector α is

$$\begin{array}{c}
 U(\alpha): \begin{array}{c} 2 \rightarrow 3 \\ \downarrow \\ 2 \rightarrow 3 \leftarrow 4 \end{array}
 \end{array}$$

We have $\text{Hom}_H(U(\alpha), M(\beta)) = 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. So (α, β, γ) is an H -admissible triple.

6.7.25. *Case 3.5.* Let $\gamma = (1, 2, 2, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 1), (0, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc}
 M(\alpha): & \begin{array}{c} 1 \leftarrow 2 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \end{array} & M(\beta): \begin{array}{c} 2 \\ \downarrow \\ 2 \rightarrow 3 \end{array}
 \end{array}$$

There are only three more locally free indecomposable H -modules with rank vector α , namely

$$\begin{array}{ccc}
 U_1(\alpha): & \begin{array}{c} 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \\ \downarrow \quad \downarrow \\ 1 \leftarrow 2 \end{array} & U_2(\alpha): \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \\ \downarrow \\ 2 \end{array} & U_3(\alpha): \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \\ \downarrow \\ 2 \rightarrow 3 \leftarrow 4 \end{array}
 \end{array}$$

One easily sees that $\text{Hom}_H(U_i(\alpha), M(\beta)) = 0$ for $i = 1, 2, 3$. Thus none of the $U_i(\alpha)$ can be a submodule of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.26. *Case 3.6.* Let $\gamma = (1, 2, 4, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 0), (1, 1, 2, 2))$. There exists a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. There exists only one locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.27. *Case 3.7.* Let $\gamma = (2, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 1, 2, 2), (1, 2, 2, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 & M(0, 1, 2, 2): & 2 \rightarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 & & 2 \rightarrow 3 \leftarrow 4 \end{array}$$

The only other locally free indecomposable H -module with rank vector α is

$$\begin{array}{c} U(\alpha): & 1 \\ & \downarrow \\ & 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \\ & \quad \downarrow \\ & \quad 2 \rightarrow 3 \leftarrow 4 \end{array}$$

Using Figure 3 we get $\text{Hom}_H(M(0, 1, 2, 2), M(\gamma)) = 0$. On the other hand we have $\text{Hom}_H(M(0, 1, 2, 2), U(\alpha)) \neq 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.28. *Case 3.8.* Let $\gamma = (1, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 2, 2, 0), (0, 1, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.29. *Case 3.9.* Let $\gamma = (1, 2, 2, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 2), (1, 1, 0, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.30. *Case 3.10.* $\gamma = (1, 1, 2, 2)$, $(\alpha, \beta) = ((1, 0, 0, 0), (0, 1, 2, 2))$. We have

$$\begin{array}{ccc} M(\gamma): & 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 & M(\beta): & 2 \rightarrow 3 \leftarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 & & 2 \rightarrow 3 \leftarrow 4 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.31. *Case 4.1.* Let $\gamma = (1, 1, 2, 1)$ and $(\alpha, \beta) = ((0, 0, 0, 1), (1, 1, 2, 0))$. We have

$$\begin{array}{ccc} M(\gamma): & 1 \leftarrow 2 \leftarrow 3 & M(\beta): & 1 \leftarrow 2 \leftarrow 3 \\ & \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & & 1 \leftarrow 2 \leftarrow 3 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.32. *Case 4.2.* Let $\gamma = (1, 1, 1, 1)$ and $(\alpha, \beta) = ((0, 0, 0, 1), (1, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.33. *Case 4.3.* Let $\gamma = (1, 2, 2, 1)$ and $(\alpha, \beta) = ((1, 1, 1, 0), (0, 1, 1, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \leftarrow 2 \leftarrow 3 & \\ & \downarrow & \downarrow \\ & 1 \leftarrow 2 & \end{array} \quad \begin{array}{ccc} M(\beta): & 2 \leftarrow 3 \rightarrow 4 & \\ & \downarrow & \\ & 2 & \end{array}$$

There is only one more locally free indecomposable H -module with dimension vector β , namely

$$\begin{array}{ccc} V(\beta): & 2 & \\ & \downarrow & \\ & 2 \leftarrow 3 \rightarrow 4 & \end{array}$$

We have $\text{Hom}_H(V(\beta), E_2) \neq 0$. By Figure 4 we have $\text{Hom}_H(M(\gamma), E_2) = 0$. Thus $V(\beta)$ cannot be a factor module of $M(\gamma)$. This shows that (α, β, γ) is an H -admissible triple.

6.7.34. *Case 4.4.* Let $\gamma = (1, 2, 3, 2)$ and $(\alpha, \beta) = ((0, 1, 1, 1), (1, 1, 2, 1))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 2 \leftarrow 3 \rightarrow 4 & \\ & \downarrow & \\ & 2 & \end{array} \quad \begin{array}{ccc} M(\beta): & 1 \leftarrow 2 \leftarrow 3 & \\ & \downarrow & \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & \end{array}$$

There is only one other locally free indecomposable H -module with rank vector α , namely

$$\begin{array}{ccc} U(\alpha): & 2 & \\ & \downarrow & \\ & 2 \leftarrow 3 \rightarrow 4 & \end{array}$$

We have $\text{Hom}_H(U(\alpha), M(\beta)) = 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. Thus (α, β, γ) is an H -admissible triple.

6.7.35. *Case 4.5.* Let $\gamma = (1, 2, 3, 1)$ and $(\alpha, \beta) = ((1, 1, 2, 1), (0, 1, 1, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\beta): & 2 \leftarrow 3 & \\ & \downarrow & \\ & 2 & \end{array}$$

There is only one other locally free indecomposable H -module with rank vector β , namely

$$\begin{array}{ccc} V(\beta): & 2 & \\ & \downarrow & \\ & 2 \leftarrow 3 & \end{array}$$

By Figure 4 we have $\text{Hom}_H(M(\gamma), E_2) = 0$. We have $\text{Hom}_H(V(\beta), E_2) \neq 0$. Thus $V(\beta)$ cannot be a factor module of $M(\gamma)$. So (α, β, γ) is an H -admissible triple.

6.7.36. *Case 4.6.* Let $\gamma = (1, 2, 2, 2)$ and $(\alpha, \beta) = ((0, 1, 0, 0), (1, 1, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The only locally free indecomposable H -module with rank vector α is $M(\alpha)$. Thus (α, β, γ) is an H -admissible triple.

6.7.37. *Case 4.7.* Let $\gamma = (2, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 1, 2, 2), (1, 2, 2, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. We have

$$\begin{array}{ccc} M(\alpha): & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & M(0, 1, 1, 0): & 2 \leftarrow 3 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & & 2 \end{array}$$

There is only one other locally free indecomposable H -module with rank vector α , namely

$$\begin{array}{c} U(\gamma): & 1 \\ & \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \\ & \quad \downarrow \\ & \quad 2 \leftarrow 3 \rightarrow 4 \end{array}$$

We see from Figure 4 that $\text{Hom}_H(M(0, 1, 1, 0), M(\gamma)) = 0$. On the other hand we have $\text{Hom}_H(M(0, 1, 1, 0), U(\alpha)) \neq 0$. Thus $U(\alpha)$ cannot be a submodule of $M(\gamma)$. So (α, β, γ) is an H -admissible triple.

6.7.38. *Case 4.8.* Let $\gamma = (1, 3, 4, 2)$ and $(\alpha, \beta) = ((1, 2, 2, 0), (0, 1, 2, 2))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\beta)$ is the only locally free indecomposable H -module with rank vector β . Thus (α, β, γ) is an H -admissible triple.

6.7.39. *Case 4.9.* Let $\gamma = (1, 2, 4, 2)$ and $(\alpha, \beta) = ((0, 1, 2, 2), (1, 1, 2, 0))$. There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.7.40. *Case 4.10.* Let $\gamma = (1, 1, 2, 2)$ and $(\alpha, \beta) = ((1, 0, 0, 0), (0, 1, 2, 2))$. We have

$$\begin{array}{ccc} M(\gamma): & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & M(\beta): & 2 \leftarrow 3 \rightarrow 4 \\ & \downarrow \quad \downarrow & & \downarrow \\ & 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 & & 2 \leftarrow 3 \rightarrow 4 \end{array}$$

There obviously exists a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) The module $M(\alpha)$ is the only locally free indecomposable H -module with rank vector α . Thus (α, β, γ) is an H -admissible triple.

6.8. **Type G_2 .** Let

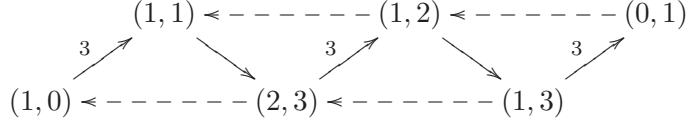
$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

be a Cartan matrix of type G_2 . Then $D = \text{diag}(3, 1)$ is the minimal symmetrizer of C . Let $\Omega = \{(1, 2)\}$. (The case $\Omega = \{(2, 1)\}$ is done dually.) The algebra $H = H(C, D, \Omega)$ is defined by the quiver

$$\begin{array}{c} \varepsilon_1 \\ \curvearrowright \\ 1 \xleftarrow{\alpha_{12}} 2 \end{array}$$

with relation $\varepsilon_1^3 = 0$. The Auslander-Reiten quiver of the corresponding modulated graph of type G_2 is shown in Figure 5. The two subgraphs of the form

$$\begin{array}{ccc} & \gamma & \\ \nearrow 3 & & \searrow \\ \alpha & & \beta \end{array}$$


 FIGURE 5. Type G_2

stand for Auslander-Reiten sequences

$$0 \rightarrow X(\alpha) \rightarrow X(\gamma)^3 \rightarrow X(\beta) \rightarrow 0.$$

As before, for the simple roots α_i , $i = 1, 2$ we define $\theta_{\alpha_i} := \theta_i$. Now we treat the remaining four positive roots $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 3)$.

6.8.1. *Case 1.1.* Let $\gamma = (1, 1)$ and $(\alpha, \beta) = ((1, 0), (0, 1))$. We have

$$\begin{array}{c} M(\gamma): \quad 1 \longleftarrow 2 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \end{array}$$

There is a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. (This is not a pseudo AR-sequence.) Set $\theta_\gamma := [\theta_\alpha, \theta_\beta]$. It follows that $\theta_\gamma(M(\gamma)) = 1$.

6.8.2. *Case 1.2.* Let $\gamma = (1, 2)$ and $(\alpha, \beta) = ((1, 0), (0, 1))$. We have

$$\begin{array}{c} M(\gamma): \quad 1 \longleftarrow 2 \\ \downarrow \\ 1 \longleftarrow 2 \\ \downarrow \\ 1 \end{array}$$

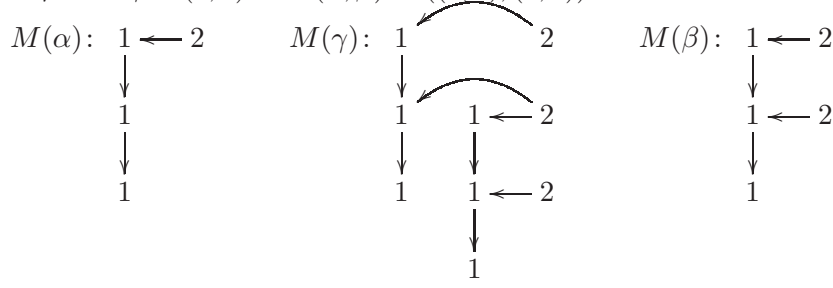
There is a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta)^2 \rightarrow 0$. (This is not a pseudo AR-sequence.) It is straightforward to check that $M(\gamma)$ does not have any filtrations of type (β, β, α) or (β, α, β) . Set $\theta_\gamma := 1/2[[\theta_\alpha, \theta_\beta], \theta_\beta]$. It follows that $\theta_\gamma(M(\gamma)) = 1$.

6.8.3. *Case 1.3.* Let $\gamma = (1, 3)$ and $(\alpha, \beta) = ((1, 0), (0, 1))$. We have

$$\begin{array}{c} M(\gamma): \quad 1 \longleftarrow 2 \\ \downarrow \\ 1 \longleftarrow 2 \\ \downarrow \\ 1 \longleftarrow 2 \end{array}$$

There is a short exact sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta)^3 \rightarrow 0$. (This is not a pseudo AR-sequence.) Obviously, $M(\gamma)$ does not have any filtrations of type $(\beta, \beta, \beta, \alpha)$, $(\beta, \beta, \alpha, \beta)$ or $(\beta, \alpha, \beta, \beta)$. Set $\theta_\gamma := 1/6[[[\theta_\alpha, \theta_\beta], \theta_\beta], \theta_\beta]$. It follows that $\theta_\gamma(M(\gamma)) = 1$.

6.8.4. *Case 1.4.* Let $\gamma = (2, 3)$ and $(\alpha, \beta) = ((1, 1), (1, 2))$. We have



There is a pseudo AR-sequence $0 \rightarrow M(\alpha) \rightarrow M(\gamma) \rightarrow M(\beta) \rightarrow 0$. Using Figure 5 we get that $\text{Ext}_H^1(M(\beta), M(\alpha))$ is 1-dimensional. Thus up to equivalence there exists only one non-split extension of the above form. In contrast to all other cases studied before, we have $\text{Hom}_H(M(\alpha), M(\beta)) \neq 0$ and there is more than one submodule of $M(\gamma)$ which is isomorphic to $M(\alpha)$. So we need to use a different strategy for this exceptional case. We have $\theta_\alpha = [\theta_1, \theta_2]$ and $\theta_\beta = 1/2[[\theta_1, \theta_2], \theta_2]$. Now define $\theta_\gamma := 1/2[\theta_\alpha, \theta_\beta]$. We get

$$\theta_\gamma = 1/4([\theta_1, \theta_2] * [[\theta_1, \theta_2], \theta_2] - [[\theta_1, \theta_2], \theta_2] * [\theta_1, \theta_2]).$$

Expanding this, we get

$$\begin{aligned} \theta_\gamma = & 1/4(\theta_1 * \theta_2 * \theta_1 * \theta_2 * \theta_2 - 3(\theta_1 * \theta_2 * \theta_2 * \theta_1 * \theta_2) \\ & + 2(\theta_1 * \theta_2 * \theta_2 * \theta_2 * \theta_1) + (\text{a sum of monomials starting with } \theta_2)). \end{aligned}$$

Let $\Phi = (\beta_1, \dots, \beta_5)$ with $\beta_i \in \{\alpha_1, \alpha_2\}$. Then one easily checks that $M(\gamma)$ has a filtration of type Φ only if $\Phi = (\alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_2)$ or $\Phi = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_2)$. Thus we have

$$\theta_\gamma(M(\gamma)) = 1/4(\theta_1 * \theta_2 * \theta_1 * \theta_2 * \theta_2)(M(\gamma)).$$

Now one calculates that $\theta_\gamma(M(\gamma)) = 1$.

7. PROOF OF THE MAIN RESULT

7.1. The main result. The following theorem is our main result. It is an analogue of [S, Theorem 4.7] for our algebra $H(C, D, \Omega)$, where C is of Dynkin type.

Theorem 7.1. *Let $H = H(C, D, \Omega)$, $\mathcal{M} = \mathcal{M}(H)$ and $\mathfrak{n} = \mathfrak{n}(C)$. Assume that C is of Dynkin type. Then the following hold:*

- (i) *The Lie algebra $\mathcal{P}(\mathcal{M})$ is isomorphic to \mathfrak{n} .*
- (ii) *The homomorphism $\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}$ is an isomorphism of Hopf algebras.*

Conjecture 7.2. *Theorem 7.1 remains true for all symmetrizable generalized Cartan matrices C and all symmetrizers D of C .*

We tried to prove Conjecture 7.2 by adapting Schofield's [S] approach to our setup. However we couldn't find an analogue for some delicate steps in his proof.

7.2. Proof of Theorem 7.1. Let C be a symmetrizable generalized Cartan matrix, and let D be a symmetrizer of C . Furthermore, let Ω be an orientation of C . For C symmetric and D the identity matrix we know already from [S] that $\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}$ is an isomorphism. This covers the Dynkin types A_n , D_n , E_6 , E_7 and E_8 .

Recall from Corollary 4.10 that there is a surjective Hopf algebra homomorphism $\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}$ which maps e_i to θ_i for all i . The homomorphism η restricts to a surjective Lie algebra homomorphism $\mathfrak{n} \rightarrow \mathcal{P}(\mathcal{M})$ and to a surjective linear map $\eta_{H,\gamma}: \mathfrak{n}_\gamma \rightarrow \mathcal{P}(\mathcal{M})_\gamma$

of weight spaces. (The positive roots are the weights of \mathfrak{n} .) If C is of Dynkin type, then all weight spaces \mathfrak{n}_γ are 1-dimensional. Now for C Dynkin and D minimal, Theorem 6.1 implies that $\eta_{H,\gamma}$ is an isomorphism for all γ . Thus η_H restricts to a Lie algebra isomorphism $\mathfrak{n} \rightarrow \mathcal{P}(\mathcal{M})$. But this implies that η_H itself has to be an algebra isomorphism. This proves Theorem 7.1 for D minimal symmetrizer.

Let now D be minimal and fix some $k \geq 2$. Without loss of generality we can assume that C is connected. (This is equivalent to $Q(C, \Omega)$ being connected.) For $l \geq 2$, set $H = H(C, D, \Omega)$ and $H(l) := H(C, lD, \Omega)$. Recall that, writing $Z(l) := K[\epsilon]/(\epsilon^l)$, we have that $H(l)$ is a $Z(l)$ -algebra which is free as a $Z(l)$ -module. In [GLS2, Section 2] we construct a functor

$$R_{l.f.}(l): \text{rep}_{l.f.}(H(l)) \rightarrow \text{rep}_{l.f.}(H(l-1)), \quad M \mapsto M/\epsilon^{l-1}M, \quad (l \geq 2).$$

By [GLS2, Lemma 2.1 and Proposition 2.2] the functor $R_{l.f.}(l)$ preserves rank vectors and yields a bijection between the isomorphism classes of rigid locally free $H(l)$ -modules and the isomorphism classes of rigid locally free $H(l-1)$ -modules. Suppose $M(k)$ is a rigid locally free $H(k)$ -module, and let

$$M := (R_{l.f.}(2) \circ \cdots \circ R_{l.f.}(k))(M(k))$$

be the corresponding rigid locally free H -module. Then we have $\chi(\mathcal{FL}_{M(k), \mathbf{i}}) = \chi(\mathcal{FL}_{M, \mathbf{i}})$ for all sequences \mathbf{i} , see [GLS2, Corollary 1.3].

We proved already that we have a Hopf algebra isomorphism $\eta_H: U(\mathfrak{n}) \rightarrow \mathcal{M}(H)$. We also have a surjective Hopf algebra homomorphism $\eta_{H(k)}: U(\mathfrak{n}) \rightarrow \mathcal{M}(H(k))$ by Corollary 4.10. This yields a surjective Hopf algebra homomorphism $\psi: \mathcal{M}(H) \rightarrow \mathcal{M}(H(k))$ sending $\theta_{\mathbf{i}} := \theta_{i_1} \cdots \theta_{i_t}$ to $\theta'_{\mathbf{i}} := \theta'_{i_1} \cdots \theta'_{i_t}$ with $\theta'_i := \psi(\theta_i) = \eta_{H(k)}(e_i)$ for all sequences $\mathbf{i} = (i_1, \dots, i_t)$ with $1 \leq i_j \leq n$ and $t \geq 1$.

$$\begin{array}{ccc} U(\mathfrak{n}) & \xrightarrow{\eta_H} & \mathcal{M}(H) \\ \eta_{H(k)} \downarrow & \swarrow \psi & \\ \mathcal{M}(H(k)) & & \end{array}$$

Let $\alpha \in \Delta^+(C)$ be a positive root, and let $M_H(\alpha)$ (resp. $M_{H(k)}(\alpha)$) be the corresponding indecomposable rigid locally free H -module (resp. $H(k)$ -module) with rank vector α . By Theorem 6.1 there exists an element $\theta_\alpha \in \mathcal{P}(\mathcal{M}(H))_\alpha \subset \mathcal{M}(H)$ such that $\theta_\alpha(M_H(\alpha)) = 1$. Define $\theta'_\alpha := \psi(\theta_\alpha)$. We know that θ_α is of the form

$$\theta_\alpha = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \theta_{\mathbf{i}}$$

for some $\lambda_{\mathbf{i}} \in K$. Recalling (4.1), we get

$$\begin{aligned} \theta_\alpha(M_H(\alpha)) &= \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \theta_{\mathbf{i}}(M_H(\alpha)) = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \chi(\mathcal{FL}_{M_H(\alpha), \mathbf{i}}) \\ &= \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \chi(\mathcal{FL}_{M_{H(k)}(\alpha), \mathbf{i}}) = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \theta'_{\mathbf{i}} = \theta'_\alpha(M_{H(k)}(\alpha)). \end{aligned}$$

This implies that θ'_α is a non-zero element in the root space $\mathcal{P}(\mathcal{M}(H(k)))_\alpha$. This finishes the proof.

8. EXAMPLES

8.1. **A PBW-basis for Dynkin type B_2 .** Let

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

with symmetrizer $D = \text{diag}(2, 1)$ and $\Omega = \{(1, 2)\}$. Thus C is a Cartan matrix of Dynkin type B_2 . We have $f_{12} = 1$ and $f_{21} = 2$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{array}{c} \varepsilon_1 \\ \curvearrowright \\ 1 \xleftarrow{\alpha_{12}} 2 \end{array}$$

with relation $\varepsilon_1^2 = 0$. There are 5 isomorphism classes of indecomposable locally free H -modules, namely

$$\begin{array}{ccccc} E_1 = P_1: & \begin{array}{c} 1 \\ \downarrow \\ 1 \end{array} & P_2: & \begin{array}{c} 1 \leftarrow 2 \\ \downarrow \\ 1 \end{array} & I_1: & \begin{array}{c} 1 \leftarrow 2 \\ \downarrow \\ 1 \leftarrow 2 \end{array} & E_2 = I_2: & \begin{array}{c} 2 \\ \downarrow \\ 1 \leftarrow 2 \end{array} & X: & \begin{array}{c} 1 \\ \downarrow \\ 1 \leftarrow 2 \end{array} \end{array}$$

Note that P_2 and X have the same rank vector. We have

$$\begin{aligned} \theta_{(1,0)} &= \theta_1 = \mathbf{1}_{E_1}, & \theta_{(0,1)} &= \theta_2 = \mathbf{1}_{E_2}, \\ \theta_{(1,1)} &= [\theta_1, \theta_2] = \mathbf{1}_{P_2} + \mathbf{1}_X, & \theta_{(1,2)} &= 1/2[[\theta_1, \theta_2], \theta_2] = \mathbf{1}_{I_1}. \end{aligned}$$

The enveloping algebra $\mathcal{M}(H) \cong U(\mathfrak{n})$ has a Poincaré-Birkhoff-Witt basis given by

$$\theta_{(0,1)}^a * \theta_{(1,2)}^b * \theta_{(1,1)}^c * \theta_{(1,0)}^d = a!b!c!d! \sum_{k=0}^c \mathbf{1}_{I_2^a \oplus I_1^b \oplus P_2^k \oplus X^{c-k} \oplus P_1^d}, \quad (a, b, c, d \in \mathbb{Z}_{\geq 0}).$$

8.2. **Pseudo AR-sequences for Dynkin type B_2 .** Let $H = H(C, D, \Omega)$ as in Section 8.1. The Auslander-Reiten quiver of H is shown in Figure 6. In the last two rows the two modules on the left have to be identified with the corresponding two modules on the right. The numbers stand again for basis vectors, the arrows α_{12} and ε_1 of $Q(C, \Omega)$ act in the obvious way, compare Section 8.1. There are two AR-sequences with preprojective

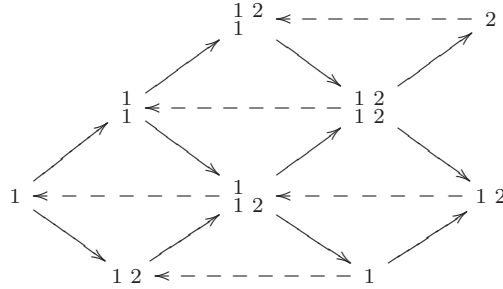


FIGURE 6. The Auslander-Reiten quiver of $H(C, D, \Omega)$ of type B_2 with D minimal.

end terms, namely $0 \rightarrow P_2 \rightarrow I_1 \rightarrow I_2 \rightarrow 0$ and $0 \rightarrow P_1 \rightarrow P_2 \oplus X \rightarrow I_1 \rightarrow 0$. The first sequence is also a pseudo AR-sequences, whereas the second sequence is not. The pseudo AR-sequence starting in P_1 is of the form $0 \rightarrow P_1 \rightarrow P_2 \oplus P_2 \rightarrow I_1 \rightarrow 0$.

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CHRISTOF GEISS
INSTITUTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
CIUDAD UNIVERSITARIA
04510 MÉXICO D.F.
MÉXICO

E-mail address: `christof@math.unam.mx`

BERNARD LECLERC
NORMANDIE UNIV, FRANCE
UNICAEN, LMNO, F-14032 CAEN FRANCE
CNRS, UMR 6139, F-14032 CAEN FRANCE
INSTITUT UNIVERSITAIRE DE FRANCE

E-mail address: `bernard.leclerc@unicaen.fr`

JAN SCHRÖER
MATHEMATISCHES INSTITUT
UNIVERSITÄT BONN
ENDENICHER ALLEE 60
53115 BONN
GERMANY

E-mail address: `schroer@math.uni-bonn.de`